A shape theorem for an epidemic model in dimension $d \geq 3$

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Abstract. We prove a shape theorem for the set of infected individuals in a spatial epidemic model with 3 states (susceptible-infected-recovered) on $\mathbb{Z}^d, d \geq 3$, when there is no extinction of the infection. For this, we derive percolation estimates (using dynamic renormalization techniques) for a locally dependent random graph in correspondence with the epidemic model.

1. Introduction

Mollison (1977, 1978) has introduced a stochastic spatial epidemic model on $\mathbb{Z}^d$ called "general epidemic model", describing the evolution of individuals submitted to infection by contact contamination of infected neighbors. More precisely, on each site of $\mathbb{Z}^d$ there is an individual who can be healthy, infected, or immune. At time 0, there is an infected individual at the origin, and all other sites are occupied by healthy individuals. Each infected individual emits germs according to a Poisson process, it stays infected for a random time, then it recovers and becomes immune to further infection. A germ emitted from $x \in \mathbb{Z}^d$ goes to one of the neighbors $y \in \mathbb{Z}^d$ of x chosen at random. If the individual at y is healthy then it becomes infected and begins to emit germs; if this individual is infected or immune, nothing happens. The germ emission processes and the durations of infections of different individuals are mutually independent.

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After Mollison's papers, this epidemic model has given rise to many studies, and other models that are variations of this "SIR" (Susceptible-Infected-Recovered) structure have been introduced. A first direction to study such models is whether the different states asymptotically survive or not, according to the values of the involved parameters (e.g. the infection and recovery rates). A second direction is the obtention of a shape theorem for the asymptotic behavior of infected individuals, when there is no extinction of the infection (throughout this paper, "extinction" is understood as "extinction of the infection").

Kelly (1977) proved that for $d = 1$, extinction is almost sure for the general epidemic model. Kuulasmaa (1982) has studied the threshold behavior of this model in dimension $d \geq 2$. He proved that the process has a critical infection rate below which extinction is almost certain, and above which there is survival, thus closing this question. His work (as well as the following ones on this model) is based on the analysis of an oriented percolation model, that he calls a "locally dependent random graph", in correspondence with the epidemic model. See also the related paper Kuulasmaa and Zachary (1984).

In the general epidemic model on $\mathbb{Z}^2$, when there is no extinction, Cox and Durrett (1988) have derived a shape theorem for the set of infected or immune individuals when the contamination rule is nearest neighbor, and the durations of infection are positive with a positive probability. A second moment is required for those durations only to localize the infected but not immune individuals within the shape obtained. This result was extended to a finite range contamination rule by Zhang (1993). The proofs in Cox and Durrett (1988); Zhang (1993) are based on the correspondence with the locally dependent random graph; they refer to Cox and Durrett (1981), which deals with first passage percolation (see also Kesten, 1986), including the possibility of infinite passage times. They rely on circuits to delimit and control open paths. This technique cannot be used for dimension greater than 2.

There was no investigation of the shape theorem for the general epidemic model in higher dimensions, until Chabot (1998, unpublished) proved it for a nearest neighbor contamination rule in dimension $d \geq 3$, with the restriction to deterministic durations of infection: in that case the oriented percolation model is comparable to a non-oriented Bernoulli percolation model (as noticed in Kuulasmaa, 1982, the case with constant durations of infection in the epidemic model is the only one where the edges are independent in the percolation model). Analyzing the epidemic model for $d \geq 3$ required heavier techniques than before: Chabot (1998) used results from Antal and Pisztora (1996) and Grimmett and Marstrand (1990) for non-oriented Bernoulli percolation to derive, for the percolation model, exponential estimates in the subcritical case on the one hand, and estimates using percolation on slabs on the other hand. To apply those results to the epidemic model required to find an alternative, in the percolation model, to the neighborhoods (for points in $\mathbb{Z}^2$) delimited by circuits of Cox and Durrett (1988). Chabot (1998) introduced new types of random neighborhoods characterized by the properties of the percolation model in dimension $d \geq 3$.

In the present work, we complete the derivation of the shape theorem for the set of infected or immune individuals in the general epidemic model with a nearest neighbor contamination rule in dimension $d \geq 3$, by proving it for random durations of infection, which are positive with a positive probability. There, the comparison with non oriented percolation done in Chabot (1998) is no longer valid, and we have to deal with an oriented dependent percolation model, with possibly infinite passage times. Our approach consists in adapting the dynamic renormalization techniques of Grimmett and Marstrand (1990) without calling on Antal and Pisztora (1996). This simplifies the paper, but we obtain sub-exponential estimates (which suffice for our purposes), instead of exponential estimates as in the paper Chabot (1998). With this in hand, it is then possible to catch hold of the skeleton of the latter: We take advantage of the random neighborhoods introduced there (they turn out to be still valid in our setting) to derive the shape theorem. Similarly to Cox and Durrett (1988), we require a moment of order d of the durations of infection only to localize the infected but not immune individuals within the shape obtained.

Let us mention two recent works, Cerf and Théret (2014) and Mourrat (2012), on shape theorems for (or related to) first passage (non dependent) percolation on $\mathbb{Z}^d$ with various assumptions on the passage times, for which the approach in Kesten (1986) is extended.

Our paper is organized as follows. In Section 2 we define the general epidemic model, the locally dependent random graph, we explicit their link, and we state the shape theorem (Theorem 2.2). Section 3 is devoted to the necessary percolation estimates on the locally dependent random graph needed for Theorem 2.2. We prove the latter in Section 4, thanks to an analysis of the travel times for the epidemic. In Appendix A, we prove all the results of Section 3 requiring dynamical renormalisation techniques.

2. The set-up: definitions and results

Let $d \geq 3$. The epidemic model on $\mathbb{Z}^d$ is represented by a Markov process $(\eta_t)_{t\geq 0}$ of state space $\Omega = \{0, i, 1\}^{\mathbb{Z}^d}$. The value $\eta_t(x) \in \{0, i, 1\}$ is the state of the individual located at site x at time t: state 1 if the individual is healthy (but not immune), state i if it is infected, or state 0 if it is immune. We will shorten this in "site x is healthy, infected or immune". We assume that at time 0, the origin $o = (0, \dots, 0)$ is the only infected site while all other sites are healthy. That is, the initial configuration η_0 is given by

$$\eta_0(o) = i, \qquad \forall\, z \neq o,\, \eta_0(z) = 1. \tag{2.1}$$

We now describe how the epidemic propagates, then we introduce a related locally dependent oriented bond percolation model on $\mathbb{Z}^d$, and finally we link the two models. We assume that all the processes and random variables we deal with are defined on a common probability space, whose probability is denoted by P, and the corresponding expectation by E.

For $x = (x_1, \dots, x_d) \in \mathbb{Z}^d, y = (y_1, \dots, y_d) \in \mathbb{Z}^d$, $\|x - y\|_1 = \sum_{i=1}^d |x_i - y_i|$ denotes the l^1 norm of $x - y$, and we write $x \sim y$ if x, y are neighbors, that is $\|x - y\|_1 = 1$. Let $(T_x, e(x, y) : x, y \in \mathbb{Z}^d, x \sim y)$ be independent random variables such that

1) the T_x's are nonnegative with a common distribution satisfying $P(T_x = 0) < 1$;
2) the $e(x, y)$'s are exponentially distributed with a parameter $\lambda > 0$.
We stress that the only assumption on the T_x's is that their distribution is not a Dirac mass on 0. They could be infinite, or without any finite moment. We define

$$X(x, y) = \begin{cases} 1 & \text{if } e(x, y) < T_x; \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

In the epidemic model, for a given infected individual x, T_x denotes the amount of time x stays infected; during this time of infection, x emits germs according to a Poisson process of parameter $2d\lambda$; when T_x is over, x recovers and its state becomes 0 forever. An emitted germ from x at some time t reaches y (say), one of the $2d$ neighbors of x, uniformly. If this neighbor y is in state 1 at time t^-, it immediately changes to state i at time t, from t begins the duration of infection T_y, and y begins to emit germs according to the same rule as x did; if this neighbor y is in state 0 or i at time t^-, nothing happens.

In the percolation model, for $x, y \in \mathbb{Z}^d, x \sim y$, the oriented bond (x, y) is said to be *open with passage time* $e(x, y)$ (abbreviated λ-*open*, or *open* when the parameter is fixed) if $X(x, y) = 1$ and *closed* (with infinite passage time) if $X(x, y) = 0$. As in Kuulasmaa (1982), we call this oriented percolation model a *locally dependent random graph*. Indeed the fact that *any* of the bonds exiting from site x is open depends on the r.v. T_x.

For $x, y \in \mathbb{Z}^d$ (not necessarily neighbors), "$x \to y$" means that there exists (at least) *an open path* from x to y, that is a path of open oriented bonds, $\Gamma_{x,y} = (z_0 = x, z_1, \ldots, z_n = y)$.

If $x \to y$, $x \neq y$, we define the *passage time on* $\Gamma_{x,y}$ to be (see (2.2))

$$\overline{\tau}(\Gamma_{x,y}) = \sum_{j=0}^{n-1} e(z_j, z_{j+1}) \tag{2.3}$$

and, if $x = y$, we put $\overline{\tau}(\Gamma_{x,x}) = 0$.
We then define the *travel time from x to y* to be

$$\tau(x, y) = \begin{cases} \inf\limits_{\{\Gamma_{x,y}\}} \overline{\tau}(\Gamma_{x,y}) & \text{if } x \neq y, x \to y, \\ 0 & \text{if } x = y, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.4}$$

where the infimum is over all possible open paths from x to y.

Coming back to the epidemic model, note that when the initial configuration is η_0 defined in (2.1), for a given site z, $\tau(o, z)$ is the duration for the infection to propagate from o to z, changing successively the values on all the sites of the involved path $\Gamma_{o,z}$ from 1 to i.

To link the two models, we define, for $t \geq 0$,

$$\Xi_t = \{x \in \mathbb{Z}^d : x \text{ is immune at time } t\} = \{x \in \mathbb{Z}^d : \eta_t(x) = 0\}; \tag{2.5}$$
$$\Upsilon_t = \{x \in \mathbb{Z}^d : x \text{ is infected at time } t\} = \{x \in \mathbb{Z}^d : \eta_t(x) = i\}. \tag{2.6}$$

We have, for $z \in \mathbb{Z}^d$, $t \geq 0$,

$$z \in \Upsilon_t \cup \Xi_t \quad \text{if and only if} \quad \tau(o,z) \leq t. \tag{2.7}$$

Indeed, $\tau(o,z) \leq t$ means that the infection has reached site z before time t, so that site z is either still infected or already immune at time t, that is $z \in \Upsilon_t \cup \Xi_t$. Conversely, if z is infected or immune at time t, it means that it has already been infected.

In the epidemic model, we denote by C_o^{out} the set of sites that will ever become infected, that is

$$C_o^{\text{out}} = \{x \in \mathbb{Z}^d : \exists\, t \geq 0, \eta_t(x) = i | \eta_0(o) = i, \forall\, z \neq o, \eta_0(z) = 1\}. \tag{2.8}$$

Then, by (2.7), C_o^{out} is the set of sites that can be reached from the origin following an open path in the percolation model. See also Cox and Durrett (1988, (1.2)), Mollison (1977, p. 322) and Kuulasmaa (1982, Lemma 3.1).

More generally, in the percolation model, for each $x \in \mathbb{Z}^d$ we define the *ingoing and outgoing clusters to and from* x to be

$$C_x^{\text{in}} = \{y \in \mathbb{Z}^d : y \to x\}, \qquad C_x^{\text{out}} = \{y \in \mathbb{Z}^d : x \to y\}, \tag{2.9}$$

and the corresponding critical values to be

$$\lambda_c^{\text{in}} = \inf\{\lambda : P(|C_x^{\text{in}}| = +\infty) > 0\}, \quad \lambda_c^{\text{out}} = \inf\{\lambda : P(|C_x^{\text{out}}| = +\infty) > 0\}, \tag{2.10}$$

where $|A|$ denotes the cardinality of a set A.

In Section 3, we will first prove the following proposition about these critical values.

Proposition 2.1. *We have $\lambda_c^{\text{in}} = \lambda_c^{\text{out}}$. This common value will be denoted by $\lambda_c = \lambda_c(\mathbb{Z}^d)$.*

Assuming that $\lambda > \lambda_c$, the most important part of our work in Section 3 will then be, thanks to dynamic renormalization techniques, to analyze for the percolation model percolation on slabs in Theorem 3.5, and, through a succession of lemmas, to establish in Proposition 3.11 subexponential estimates for the length of the shortest path between two points x and y given that $x \to y$. This will imply (see Remark 4.1) uniqueness of the infinite cluster of sites connected to $+\infty$. Proposition 3.11 contains the crucial properties we will need on the percolation model to derive our main result, the shape theorem, that we now state.

Theorem 2.2. *Assume $\lambda > \lambda_c$, and the initial configuration of the epidemic model $(\eta_t)_{t\geq 0}$ to be given by (2.1). Then there exists a convex subset $D \subset \mathbb{R}^d$ such that, for all $\varepsilon > 0$ we have, for t large enough*

$$\Big((1-\varepsilon)tD \cap C_o^{\text{out}}\Big) \subset \Big(\Xi_t \cup \Upsilon_t\Big) \subset \Big((1+\varepsilon)tD \cap C_o^{\text{out}}\Big) \ \textit{a.s.} \tag{2.11}$$

and if $E(T_o^d) < \infty$ we also have

$$\Upsilon_t \subset \Big((1+\varepsilon)tD \setminus (1-\varepsilon)tD\Big) \ \textit{a.s. for } t \textit{ large enough.} \tag{2.12}$$

In other words, the epidemic's progression follows linearly the boundary of a convex set. Note that a moment assumption on T_o is only required to localize the infected individuals, and not for the first part of the theorem, for which there is no assumption on the distribution of T_o. It is also remarkable that the fact that T_o

could be either very small or very large with respect to the exponential variables $e(x,y)$ does not play any role.

We prove Theorem 2.2 in Section 4. For this, we follow some of the fundamental steps of Cox and Durrett (1988), but since in dimensions three or higher, circuits are not useful as in dimension 2, we had to find other methods of proofs.

By (2.7), we have to analyze travel times to prove Theorem 2.2. On the percolation model, we first construct, in Section 4.1, for each site $z \in \mathbb{Z}^d$ a random neighborhood $\mathcal{V}(z)$ in such a way that two neighborhoods are always connected by open paths (these neighborhoods have to be different from those delimited by circuits of Cox and Durrett, 1988). For $z, y \in \mathbb{Z}^d$, we show that the travel time $\tau(z,y)$ is 'comparable' (in a sense precised in Lemma 4.8) to the travel time $\widehat{\tau}(z,y)$ to go from $\mathcal{V}(z)$ to $\mathcal{V}(y)$. Then we approximate the travel time between sites by a subadditive process, and we derive (in Theorem 4.12 and Section 4.3) a radial limit $\mu(x)$ (for all x), which is asymptotically the linear growth speed of the epidemic in direction x. In Theorem 4.17 we control how $\widehat{\tau}(o,\cdot)$ grows. Finally we prove in Theorem 4.18 an asymptotic shape theorem for $\widehat{\tau}(o,\cdot)$, from which we deduce Theorem 2.2.

3. Percolation estimates

In this section we collect some results concerning the locally dependent random graph, given by the random variables $(X(x,y), x, y \in \mathbb{Z}^d)$ introduced in (2.2). Our goal is to derive subexponential estimates in Proposition 3.11.

Remark 3.1. Although the r.v.'s $(X(x,y), x, y \in \mathbb{Z}^d)$ are not independent, if we denote by $(\mathrm{e}_1, \dots, \mathrm{e}_d)$ the canonical basis of $\mathbb{Z}^d$, then the random vectors $\{X(x, x+\mathrm{e}_1), \dots, X(x, x+\mathrm{e}_d), X(x, x-\mathrm{e}_1), \dots, X(x, x-\mathrm{e}_d) : x \in \mathbb{Z}^d\}$ (in which each component depends on T_x) are i.i.d., since two different vectors for $z, y \in \mathbb{Z}^d$ depend respectively on T_z and T_y which are independent. This small dependence forces us to explain why and how some results known for independent percolation remain valid in this context.

Remark 3.2. The function $X(x,y)$ is increasing in the independent random variables T_x and $-e(x,y)$. It then follows that the r.v.'s $(X(x,y) : x, y \in \mathbb{Z}^d, y \sim x)$ satisfy the following property:

(FKG) Let U and V be bounded measurable increasing functions of the random variables $(X(x_j, y_j) : x_j, y_j \in \mathbb{Z}^d, y_j \sim x_j, j \in \mathbb{N})$, then $E(UV) \geq E(U)E(V)$.

For the proof of this property, we refer to Cox and Durrett (1988, Lemma (2.1)) with the help of Harris (1960, Lemma 4.1 and its Corollary) if U and V depend on a finite number of variables $X(x_j, y_j)$, and to Grimmett (1999, Chapter 2) to take the limit for an infinite number of variables.

We will use this property in the proofs of Theorem 3.5, Lemma 3.10 and Lemma 4.6 below for U, V indicator functions involving open paths without loops, thus we will speak of increasing events rather than increasing functions.

For $n \in \mathbb{N} \setminus \{0\}$, let $B(n) = [-n, n]^d$, let $\partial B(n)$ denote the *inner vertex boundary* of $B(n)$, that is

$$\partial B(n) = \{x \in \mathbb{Z}^d : x \in B(n), x \sim y \text{ for some } y \notin B(n)\}; \tag{3.1}$$

and, for $x \in \mathbb{R}^d$, $B_x(n) = x + B(n)$. For $A, R \subset \mathbb{Z}^d$, "$A \to R$" means that there exists an open path $\Gamma_{x,y}$ from some $x \in A$ to some $y \in R$.

Theorem 3.3. *(i) Suppose $\lambda < \lambda_c^{\mathrm{out}}$, then there exists $\beta_{\mathrm{out}} > 0$ such that for all $n > 0$, $P(o \to \partial B(n)) \leq \exp(-\beta_{\mathrm{out}} n)$.*
(ii) Suppose $\lambda < \lambda_c^{\mathrm{in}}$, then there exists $\beta_{\mathrm{in}} > 0$ such that for all $n > 0$, $P(\partial B(n) \to o) \leq \exp(-\beta_{\mathrm{in}} n)$.

Theorem 3.3*(i)* is a special case of Van den Berg et al. (1998, Theorem (3.1)), whose proof can be adapted to obtain Theorem 3.3*(ii)*. It is worth noting that in the context of our paper, by Remark 3.2, Van den Berg et al. (1998, Theorem (3.1)) can be proved using the BK inequality instead of the Reimer inequality (see Grimmett, 1999, Theorems (2.12), (2.19)). Theorem 3.3 yields Proposition 2.1:

Proof of Proposition 2.1. Suppose $\lambda < \lambda_c^{\mathrm{in}}$. Then by translation invariance and Theorem 3.3*(ii)* we have that for any $x \in \partial B(n)$, $P(o \to x) \leq \exp(-\beta_{\mathrm{in}} n)$. Adding over all points of $\partial B(n)$ we get $P(o \to \partial B(n)) \leq K' n^{d-1} \exp(-\beta_{\mathrm{in}} n)$ for some constant K', which implies that $\lim_{n \to +\infty} P(o \to \partial B(n)) = 0$. Therefore $\lambda \leq \lambda_c^{\mathrm{out}}$ and $\lambda_c^{\mathrm{in}} \leq \lambda_c^{\mathrm{out}}$. The other inequality is obtained similarly. □

From now on, we assume $\lambda > \lambda_c(\mathbb{Z}^d)$ and define the following events: For $x, y \in \mathbb{Z}^d, A \subset \mathbb{Z}^d$,
(i) The event $\{x \to y \text{ within } A\}$ consists of all points in our probability space for which there exists an open path $\Gamma_{x,y} = (x_0 = x, x_1, \dots, x_n = y)$ from x to y such that $x_j \in A$ for all $j \in \{0, \dots, n-1\}$. Note that the end point y may not belong to A.
(ii) The event $\{x \to y \text{ outside } A\}$ consists of all points in our probability space for which there exists an open path $\Gamma_{x,y} = (x_0 = x, x_1, \dots, x_n = y)$ from x to y such that none of the x_j's ($j \in \{0, \dots, n\}$) belongs to A.

Definition 3.4. For $x \in \mathbb{Z}^d, A \subset \mathbb{Z}^d$ let

$$\begin{aligned} C_x^{\mathrm{in}}(A) &= \{y \in A : y \to x \text{ within } A\} \qquad \text{and} \\ C_x^{\mathrm{out}}(A) &= \{y \in A : x \to y \text{ within } A\}. \end{aligned}$$

Note that by this definition $C_x^{\mathrm{in}}(A) \subset A$ and $C_x^{\mathrm{out}}(A) \subset A$.

The rest of this section relies heavily on the techniques of Grimmett and Marstrand (1990) or Grimmett (1999, Chapter 7). We assume the reader familiar with them. We postpone to Appendix A the proofs of Theorem 3.5 and Lemma 3.7 below, which require a thoughtful adaptation of Grimmett (1999, Chapter 7) for our context of dependent percolation. Nonetheless, it is possible to go directly to Section 4, where these techniques are no longer used, assuming that Proposition 3.11 holds.

Next theorem is crucial, it states that there is percolation on slabs.

Theorem 3.5. *Assume $\lambda > \lambda_c$. For any $k \in \mathbb{N} \setminus \{0\}$, let $S_k = \{0, 1, \dots, k\} \times \mathbb{Z}^{d-1}$ denote the slab of thickness k containing o. Then for k large enough we have*

$$\inf_{x \in S_k} P(|C_x^{\mathrm{in}}(S_k)| = +\infty) > 0, \qquad \text{and} \qquad \inf_{x \in S_k} P(|C_x^{\mathrm{out}}(S_k)| = +\infty) > 0.$$

We introduce now some notation about the shortest path between two points x and y such that $x \to y$.

Notation 3.6. (a) For $A \subset \mathbb{Z}^d$ we define the *exterior vertex boundary* of A as:

$$\Delta_v A = \{x \in \mathbb{Z}^d : x \notin A, x \sim y \text{ for some } y \in A\}. \tag{3.2}$$

(b) If $x \to y$ let $D(x,y)$ be the smallest number of bonds required to build an open path from x to y (hence in this path there is no loop, and the $D(x,y)$ bonds are distinct). If $x \not\to y$, we put $D(x,y) = +\infty$.
(c) For $A \subset \mathbb{Z}^d$, $x \in A, y \in \Delta_v A$, "$D(x,y) < m$ within A" means that there is an open path $\Gamma_{x,y}$ using less than m bonds from x to y whose sites are all in A except y.

The end of this section provides some upper bounds for the tail of the conditional distribution of $D(x,y)$ given the event $\{x \to y\}$. We derive Proposition 3.11, required in Section 4, thanks to Lemmas 3.7, 3.9, 3.10. These estimates are not optimal and better results could be obtained by a thoughtful adaptation of the methods of Antal and Pisztora (1996). Instead of getting exponential decays in $\|x-y\|_1$ (or in n) we get exponential decays in $\|x-y\|_1^{1/d}$ (or in $n^{1/d}$). We have adopted this approach because those weaker results suffice for our purposes and are simpler to obtain, thus making our proof much easier to follow: it is possible to read our work knowing only Grimmett and Marstrand (1990) and not Antal and Pisztora (1996). Next lemma is inspired by Grimmett and Marstrand (1990, Section 5*(f)* p. 454).

Lemma 3.7. *Assume $\lambda > \lambda_c$. There exist $\delta > 0$, $k \in \mathbb{N} \setminus \{0\}$ and $\mathrm{C}_1 = \mathrm{C}_1(k) > 0$ such that*
(i) $\forall n > 0,\ x \in B(n+k) \setminus B(n),\ y \in (B(n+k) \setminus B(n)) \cup \Delta_v(B(n+k) \setminus B(n))$ we have :

$$P(x \to y \quad within \quad B(n+k) \setminus B(n)) > \delta.$$

(ii) Let for $(n,m) \in \mathbb{Z}^2$ with $n < m$, and for $\ell \geq 0$,

$$\begin{aligned} A(n,m,\ell) \quad = \quad & \{z : -k+n \leq z_1 < n, -\infty < z_2 \leq \ell + k\} \cup \\ & \{z : -k+n \leq z_1 \leq m+k, \ell < z_2 \leq \ell+k\} \cup \\ & \{z : m < z_1 \leq m+k, -\infty < z_2 \leq \ell + k\}. \end{aligned} \tag{3.3}$$

$\forall n < m,\ \forall x \in A(n,m,0), \forall y \in A(n,m,0) \cup \Delta_v A(n,m,0)$, we have:

$$P(D(x,y) < \mathrm{C}_1(\|x-y\|_1 + (-x_2)^+ + (-y_2)^+) \quad within\ A(n,m,0)) > \delta.$$

We again introduce some notation, to decompose in Lemma 3.9 a path from the center of a box to its boundary through hyperplanes.

Notation 3.8. Let k be given by Lemma 3.7 and let x and y be points in $\mathbb{Z}^d$. For $\ell \in \mathbb{Z}$ let $H_\ell = \{z \in \mathbb{Z}^d : z_1 = \ell\}$ and define the events, for $n \in \mathbb{N}$,

$$\begin{aligned} J_n \quad &= \quad \{x \to H_{x_1 - 1 - jk} \text{ within } B_x(nk), j = 0, \dots, \lfloor n/2 \rfloor\} \cap \\ & \qquad \{H_{y_1+1+jk} \to y \text{ within } B_y(nk), j = 0, \dots, \lfloor n/2 \rfloor\}, \\ G_n \quad &= \quad \{x \to \partial B_x(nk),\ \partial B_y(nk) \to y\}, \end{aligned}$$

where, for any $a \in \mathbb{R}$, $\lfloor a \rfloor$ denotes the greatest integer not greater than a.

Lemma 3.9. *Assume $\lambda > \lambda_c$. Let k be given by Lemma 3.7 and let x, y be points in $\mathbb{Z}^d$. Then, for $n \in \mathbb{N} \setminus \{0\}$ there exists $\beta > 0$ such that*

$$P(J_n | G_n) \geq 1 - \exp(-\beta n).$$

Proof of Lemma 3.9. By translation invariance we may assume that x is the origin. We start showing that for some constant $\beta' > 0$ and all n

$$P(o \to H_{-1-jk} \text{ within } B(nk), j = 0, \ldots, \lfloor n/2 \rfloor | o \to \partial B(nk)) \\ \geq 1 - \exp(-\beta' n). \tag{3.4}$$

For this we first observe that

$$\{o \to H_{-1-jk} \text{ within } B(nk) \text{ for some } \lfloor n/2 \rfloor \leq j \leq n\} \\ \subset \{o \to H_{-1-jk} \text{ within } B(nk), j = 0, \ldots, \lfloor n/2 \rfloor\}.$$

Hence (3.4) follows from

$$P(o \to H_{-1-jk} \text{ within } B(nk) \text{ for some } \lfloor n/2 \rfloor \leq j \leq n | o \to \partial B(nk)) \\ \geq 1 - \exp(-\beta' n),$$

which is a consequence of Lemma 3.7*(i)*. Since $P(\partial B_y(kn) \to y)$ is bounded below as n goes to infinity, (3.4) implies that

$$P(o \to H_{-1-jk} \text{ within } B(nk), j = 0, \ldots, \lfloor n/2 \rfloor | G_n)$$

converges to 1 exponentially fast. Similarly one proves that

$$P(H_{y_1+1+jk} \to y \text{ within } B_y(nk), j = 0, \ldots, \lfloor n/2 \rfloor | G_n)$$

converges to 1 exponentially fast, and the lemma follows. □

In Lemma 3.10 below we prove a chemical distance bound that will be used later on to derive in Remark 4.1, through Proposition 3.11, the uniqueness of the infinite cluster of sites connected to $+\infty$. The main technique is to construct an open path in a ring after independent attempts thanks on the one hand to Lemma 3.9 whose J_n's enable to get disjoint slabs, and on the other hand to Lemma 3.7*(ii)* once we find the appropriate ring.

Lemma 3.10. *Assume $\lambda > \lambda_c$. Let k be given by Lemma 3.7, and let G_n be as in Lemma 3.9. Then, there exist constants C_2, C_3 and $\alpha_2 > 0$ such that, for all $x, y \in \mathbb{Z}^d$, $n \in \mathbb{N} \setminus \{0\}$, we have*

$$P(D(x,y) > \mathrm{C}_2 \|x-y\|_1 + \mathrm{C}_3 (nk)^d | \ G_n) \leq \exp(-\alpha_2 n).$$

Proof of Lemma 3.10. Again, by translation invariance we may assume that x is the origin and without loss of generality, we also assume that $y_1 > 0$ and $y_2 \geq 0$. By Lemma 3.9 it suffices to show that

$$P(D(o,y) > \mathrm{C}_2 \|y\|_1 + \mathrm{C}_3 (nk)^d | \ J_n)$$

decays exponentially in n.

For $0 \leq j \leq \lfloor n/2 \rfloor$, let (see (3.3)) $A_j = A(-jk, y_1 + jk, y_2 + jk)$. Note that the sets $A_0, \ldots, A_{\lfloor n/2 \rfloor}$ are disjoint. Figure 1 should help the reader to visualize them. Our aim is to find paths from o to y through independent attempts, which will enable to use Lemma 3.7*(ii)* in each set A_j. This is why we have first replaced G_n by J_n to condition with.

On the event J_n, we can reach from the origin each of the sets A_i by means of an open path contained in $B(nk)$ and from each of these sets we can reach y by means of an open path contained in $B_y(nk)$. Hence, on J_n for each $j \in \{0, \ldots, \lfloor n/2 \rfloor\}$ there exist a random point $U_j \in B(nk) \cap A_j$ and an open path from o to U_j such

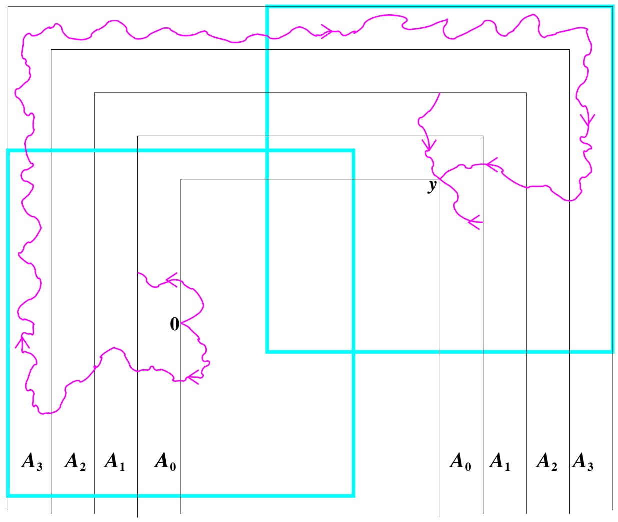

FIGURE 3.1. the event W_3

that all its sites except U_j are in $B(nk) \cap (\cap_{\ell=j}^{\lfloor n/2 \rfloor} A_\ell^c)$. If there are many possible values of U_j we choose the first one in some arbitrary deterministic order. Similarly, there is a random point $V_j \in B_y(nk) \cap \Delta_v A_j$ and an open path from V_j to y with all its sites in $B_y(nk) \cap (\cap_{\ell=j}^{\lfloor n/2 \rfloor} A_\ell^c)$. Let u^j and v^j be possible values of U_j and V_j respectively. Then let C_1 be as in Lemma 3.7 and define

$$\begin{aligned} F_j(u^j, v^j) &= \{U_j = u^j, V_j = v^j\}, \\ E_j(u^j, v^j) &= \{D(u^j, v^j) < \mathrm{C}_1(\|u^j - v^j\|_1 + |u_2^j| + |v_2^j|) \text{ within } A_j\} \text{ and} \\ W_j &= \cup_{u^j, v^j} \left(F_j(u^j, v^j) \cap E_j(u^j, v^j)\right), \end{aligned}$$

where the union is over all possible values of U_j and V_j. Now we define a subset of $\mathbb{Z}^d$

$$R_j = \Big(B(nk) \cup B_y(nk) \cup (A_0 \cup \cdots \cup A_{j-1})\Big) \cap \Big(A_j^c \cap \cdots \cap A_{n-1}^c\Big), \tag{3.5}$$

and we denote by σ_j the σ-algebra generated by $\{T_x, e(x,y) : x \in R_j, x \sim y\}$. Then, noting that $\mathbf{1}_{F_j(u^j,v^j)} \Pi_{\ell=0}^{j-1} \mathbf{1}_{W_\ell^c}$ is σ_j-measurable, write for $j = 1, \ldots, \lfloor n/2 \rfloor$:

$$\begin{aligned} P\left(W_j \cap J_n \cap (\cap_{\ell=0}^{j-1} W_\ell^c)\right) &= \sum_{u^j, v^j} E\left(\mathbf{1}_{F_j(u^j,v^j)} \mathbf{1}_{E_j(u^j,v^j)} \mathbf{1}_{J_n} (\Pi_{\ell=0}^{j-1} \mathbf{1}_{W_\ell^c})\right) \\ &= \sum_{u^j, v^j} E\left(\mathbf{1}_{F_j(u^j,v^j)} (\Pi_{\ell=0}^{j-1} \mathbf{1}_{W_\ell^c}) E(\mathbf{1}_{J_n} \mathbf{1}_{E_j(u^j,v^j)} | \sigma_j)\right) \\ &\geq \sum_{u^j, v^j} P(E_j(u^j, v^j)) E\left(\mathbf{1}_{F_j(u^j,v^j)} (\Pi_{\ell=0}^{j-1} \mathbf{1}_{W_\ell^c}) E(\mathbf{1}_{J_n} | \sigma_j)\right) \\ &= \sum_{u^j, v^j} P(E_j(u^j, v^j)) E\left(\mathbf{1}_{F_j(u^j,v^j)} (\Pi_{\ell=0}^{j-1} \mathbf{1}_{W_\ell^c}) \mathbf{1}_{J_n}\right) \\ &\geq \delta \sum_{u^j, v^j} P\left(F_j(u^j, v^j) \cap J_n \cap (\cap_{\ell=0}^{j-1} W_\ell^c)\right) = \delta P\left(J_n \cap (\cap_{\ell=0}^{j-1} W_\ell^c)\right), \end{aligned} \tag{3.6}$$

where the sums are over all possible values of U_j and V_j, the first inequality follows from Remark 3.2 since both J_n and $E_j(u^j, v^j)$ are increasing events, and from the fact that $E_j(u^j, v^j)$ is independent of σ_j; the second inequality follows from Lemma 3.7*(ii)* and the last equality from the fact that J_n is contained in the union of the $F_j(u^j, v^j)$'s which are disjoint. We rewrite (3.6) as

$$\begin{aligned} P\left(J_n \cap (\cap_{\ell=0}^{j} W_\ell^c)\right) &\leq (1-\delta) P\left(W_j \cap J_n \cap (\cap_{\ell=0}^{j-1} W_\ell^c)\right) \\ &\leq (1-\delta) P\left(J_n \cap (\cap_{\ell=0}^{j-1} W_\ell^c)\right) \end{aligned}$$

Now, proceeding by induction on j one gets:

$$P\left(J_n \cap (\cap_{\ell=0}^{\lfloor n/2 \rfloor -1} W_\ell^c)\right) \leq (1-\delta)^{\lfloor n/2 \rfloor} P(J_n).$$

Since we can choose C_2 and C_3 in such a way that the event $\{D(o,y) > \mathrm{C}_2 \|y\|_1 + \mathrm{C}_3 (nk)^d\}$ does not occur if any of the W_i's occurs, the lemma follows. $\square$

Next proposition concludes this section.

Proposition 3.11. *Assume $\lambda > \lambda_c$.*

(i) Let C_2 be as in Lemma 3.10. Then, there exists $\alpha_3 > 0$ such that for all $x, y \in \mathbb{Z}^d, n \in \mathbb{N}$, we have

$$P(D(x,y) \geq \mathrm{C}_2 \|x-y\|_1 + n^d | x \to y) \leq \exp(-\alpha_3 n);$$

(ii) $P(x \to y |\ |C_x^{\text{out}}| = +\infty,\ |C_y^{\text{in}}| = +\infty) = 1.$

Proof of Proposition 3.11. (i) Modifying the constant α_2, the statement of Lemma 3.10 above holds for $\mathrm{C}_3 = 1/k^d$.

(ii) We have that $\{x \to \infty \text{ and } \infty \to y\} = \cap_n G_n$. Hence for all k,

$$\begin{aligned} P(D(x,y) = +\infty, x \to \infty \text{ and } \infty \to y) &\leq P(D(x,y) = +\infty, G_k) \\ &\leq P(D(x,y) = +\infty | G_k), \end{aligned}$$

which converges to 0 when k goes to infinity by Lemma 3.10. We thus have $P(D(x,y) = +\infty | x \to \infty \text{ and } \infty \to y) = 0$. $\square$

4. The shape theorem

In the percolation model, let C^∞ be the cluster of sites connected to ∞:

$$C^\infty = \{x \in \mathbb{Z}^d : x \to \infty \text{ and } \infty \to x\}. \tag{4.1}$$

Remark 4.1. As a consequence of Proposition 3.11*(ii)*, C^∞ is a connected set: if two sites x, y of $\mathbb{Z}^d$ belong to C^∞, then $x \to y$ and $y \to x$.

4.1. *Neighborhoods in C^∞.* In this subsection, we construct neighborhoods $\mathcal{V}(\cdot)$ of sites in $\mathbb{Z}^d$.

We first deal separately with finite clusters, which will have no influence on the asymptotic shape of the epidemic. We will include them in the neighborhoods $\mathcal{V}(\cdot)$ of sites we construct.

Definition 4.2. For $x \in \mathbb{Z}^d$, let

$$\begin{cases} R_x^{\text{out}} = \{y \in \mathbb{Z}^d : x \to y \text{ outside } C^\infty\} & \text{(outgoing root from } x); \\ R_x^{\text{in}} = \{y \in \mathbb{Z}^d : y \to x \text{ outside } C^\infty\} & \text{(incoming root to } x). \end{cases}$$

In particular x belongs to R_x^{out} and R_x^{in} if and only if $x \notin C^\infty$. Otherwise R_x^{out} and R_x^{in} are empty. By next lemma, the distribution of the radius of $R_o^{\text{out}} \cup R_o^{\text{in}}$ decreases exponentially.

Lemma 4.3. *There exists $\sigma_1 = \sigma_1(\lambda, d) > 0$ such that, for all $n \in \mathbb{N}$,*

$$P\left((R_o^{\text{out}} \cup R_o^{\text{in}}) \cap \partial B(n) \neq \emptyset\right) \leq \exp(-\sigma_1 n).$$

Proof of Lemma 4.3. For $n \in \mathbb{N} \setminus \{0\}$, $R_o^{\text{out}} \cap \partial B(2n) \neq \emptyset$ means that there exists an open path $o \to \partial B(2n)$ outside C^∞. This implies that there exists $x \in \partial B(n)$ satisfying $o \to x \to \partial B(2n)$ outside C^∞. Similarly, $R_o^{\text{in}} \cap \partial B(2n) \neq \emptyset$ implies that there exists $x \in \partial B(n)$ satisfying $\partial B(2n) \to x \to o$ outside C^∞. Then for such a point, either the cluster C_x^{out} or the cluster C_x^{in} is finite, and has a radius larger than or equal to n. Relying on Proposition A.11,*b)* in Appendix A, we can follow the proof of Grimmett (1999, Theorems (8.18), (8.21)) to get the existence of $\sigma_0 = \sigma_0(\lambda, d) > 0$ such that:

$$\begin{cases} P(C_x^{\text{out}} \cap \partial B_x(n) \neq \emptyset, |C_x^{\text{out}}| < +\infty) \leq \exp(-\sigma_0 n); \\ P(C_x^{\text{in}} \cap \partial B_x(n) \neq \emptyset, |C_x^{\text{in}}| < +\infty) \leq \exp(-\sigma_0 n). \end{cases} \tag{4.2}$$

Hence

$$\begin{aligned} P\left((R_o^{\text{out}} \cup R_o^{\text{in}}) \cap \partial B(2n) \neq \emptyset\right) &\leq P\left(R_o^{\text{out}} \cap \partial B(2n) \neq \emptyset\right) \\ &\quad + P\left(R_o^{\text{in}} \cap \partial B(2n) \neq \emptyset\right) \\ &\leq 2 \sum_{x \in \partial B(n)} P(|C_x^{\text{out}}| < +\infty, x \to \partial B_x(n)) \\ &\quad + 2 \sum_{x \in \partial B(n)} P(|C_x^{\text{in}}| < +\infty, \partial B_x(n) \to x) \\ &\leq 4 |\partial B(n)| \exp(-\sigma_0 n) \end{aligned}$$

which induces the result. □

To define the neighborhood $\mathcal{V}(x)$ on C^∞ of a site x, we introduce the smallest box whose interior contains R_x^{out} and R_x^{in}, which contains elements of C^∞, and is such that two elements of C^∞ in this box are connected by an open path which does not exit from a little larger box. For this last condition, which will enable to bound the travel time through $\mathcal{V}(x)$, we use the parameter C_2 obtained in Lemma 3.10.

Definition 4.4. Let $\mathrm{C}' = \mathrm{C}_2 d + 2$. Let $\kappa(x)$ be the smallest $l \in \mathbb{N} \setminus \{0\}$ such that

$$\begin{cases} (i) \quad \partial B_x(l) \cap \left(R_x^{\text{out}} \cup R_x^{\text{in}}\right) = \emptyset; \\ (ii) \;\; B_x(l) \cap C^\infty \neq \emptyset; \\ (iii) \, \forall\, (y, z) \in (B_x(l) \cap C^\infty)^2, \, y \to z \text{ within } B_x(\mathrm{C}' l). \end{cases}$$

Remark 4.5. By *(i)* above, $R_x^{\text{out}} \cup R_x^{\text{in}} \subset B_x(\kappa(x))$.

In the next lemma, we bound the probability a box of size n does not admit properties *(i)–(iii)* above, that is, we prove that the random variable $\kappa(x)$ has a sub-exponential tail.

Lemma 4.6. *There exists a constant $\sigma = \sigma(\lambda, d) > 0$ such that, for any $n \in \mathbb{N}$,*

$$P(\kappa(x) \geq n) \leq \exp(-\sigma n^{1/d}).$$

Proof of Lemma 4.6. We show that the probability that any of the 3 conditions in Definition 4.4 is not achieved for n decreases exponentially in $n^{1/d}$:
(i) By translation invariance, we have by Lemma 4.3,

$$P\left(\partial B_x(n) \cap \left(R_x^{\text{out}} \cup R_x^{\text{in}}\right) \neq \emptyset\right) \leq \exp(-\sigma_1 n). \tag{4.3}$$

(ii) There exist $k \in \mathbb{N}$, $\sigma_2 = \sigma_2(\lambda, d) > 0$ such that for any $n \in \mathbb{N}$,

$$P(B_x(n) \cap C^\infty = \emptyset) \leq \exp(-\sigma_2 \lfloor n/(k+1) \rfloor). \tag{4.4}$$

Indeed, let $k = k(\lambda, d)$ be large enough for the conclusions of Theorem 3.5 to hold on the slab S_k. Then we have

$$\begin{aligned} P(B_x(n) \cap C^\infty = \emptyset) &\leq P(\forall\, z \in \{x + j\mathrm{e}_1, 0 \leq j \leq n\}, z \notin C^\infty) \\ &= P(\forall\, z \in \{x + j\mathrm{e}_1, 0 \leq j \leq n\}, C_z^{\text{in}} \text{ or } C_z^{\text{out}} \text{ is finite}) \end{aligned}$$

We denote by $S_k(l) = \{l(k+1), \cdots, (l+1)(k+1) - 1\} \times \mathbb{Z}^{d-1}$ for $l \geq 0$ the slab of thickness k to which z belongs. If C_z^{in} (or C_z^{out}) is finite, so is $C_z^{\text{in}}(S_k(l))$ (or $C_z^{\text{out}}(S_k(l))$). Because $\{|C_z^{\text{in}}(S_k(l))| = +\infty\}$ and $\{|C_z^{\text{out}}(S_k(l))| = +\infty\}$ are increasing events it follows from Theorem 3.5 and the FKG inequality (see Remark 3.2) that

$$\begin{aligned} &\inf_{u \in S_k(l)} P(|C_u^{\text{in}}(S_k(l))| = |C_u^{\text{out}}(S_k(l))| = +\infty) \\ &\geq \inf_{u \in S_k(l)} \left(P(|C_u^{\text{in}}(S_k(l))| = +\infty) P(|C_u^{\text{out}}(S_k(l))| = +\infty)\right) > 0. \end{aligned} \tag{4.5}$$

Since events occurring in two different slabs are independent, we have

$$\begin{aligned} & P(\forall\, z \in \{x + j\mathrm{e}_1, 0 \leq j \leq n\}, z \notin C^\infty) \\ \leq\ & P(\forall\, l \geq 0, \forall\, z \in \{x + j\mathrm{e}_1, 0 \leq j \leq n\} \cap S_k(l), \\ & \qquad C_z^{\text{in}}(S_k(l)) \text{ or } C_z^{\text{out}}(S_k(l)) \text{ is finite}) \\ \leq\ & (P(\forall\, z \in \{j\mathrm{e}_1, 0 \leq j \leq k\}, \\ & \qquad C_z^{\text{in}}(S_k(0)) \text{ or } C_z^{\text{out}}(S_k(0)) \text{ is finite}\big)^{\lfloor n/(k+1) \rfloor} \\ \leq\ & \exp(-\sigma_2 \lfloor n/(k+1) \rfloor) \end{aligned}$$

with $\sigma_2 = \sigma_2(\lambda, d) > 0$, independent of n, because, for $z_0 = \lfloor k/2 \rfloor \mathrm{e}_1$, using (4.5) we have

$$\begin{aligned} & P(\exists\, z \in \{x + j\mathrm{e}_1, 0 \leq j \leq k\}, |C_z^{\text{in}}(S_k(0))| = |C_z^{\text{out}}(S_k(0))| = +\infty) \\ \geq\ & \qquad P(|C_{z_0}^{\text{in}}(S_k(0))| = |C_{z_0}^{\text{out}}(S_k(0))| = +\infty) > 0. \end{aligned}$$

(iii) There exists $\sigma_3 = \sigma_3(\lambda, d) > 0$ such that

$$\begin{aligned} P\left(\exists\, (y,z) \in (B_x(n) \cap C^\infty)^2,\ y \not\to z \text{ within } (B_x(\mathrm{C}'n)\right) \\ \leq \exp(-\sigma_3 n^{1/d}). \end{aligned} \tag{4.6}$$

Indeed, if no open path from y to z (both in $B_x(n) \cap C^\infty$) is contained in $B_x(\mathrm{C}'n)$, then $D(y,z) \geq 2(\mathrm{C}'-1)n$. Given our choice of C' this implies that $D(y,z) \geq \mathrm{C}_2 \|y - z\|_1 + n$. Therefore (4.6) follows from Proposition 3.11*(i)*. ☐

We define the (site) neighborhood in C^∞ of x by

$$\mathcal{V}(x) = B_x(\kappa(x)) \cap C^\infty. \tag{4.7}$$

Remark 4.7. (a) By Definition 4.4*(ii)*, $\mathcal{V}(x) \neq \emptyset$.
(b) By Definition 4.4*(iii)*, for all y, z in $\mathcal{V}(x)$, there exists at least one open path from y to z, denoted by $\Gamma^*_{y,z}$, contained in $B_x(\mathrm{C}'\kappa(x))$. If there are several such paths we choose the first one according to some deterministic order.

We finally define an "edge" neighborhood $\overline{\Gamma}(x)$ of x:

$$\begin{aligned}\overline{\Gamma}(x) &= \{(y',z') \subset B_x(\kappa(x)), (y',z') \text{ open}\} \cup \\ &\qquad \{(y',z') \in \Gamma^*_{y,z}, y,z \in \mathcal{V}(x)\}.\end{aligned} \tag{4.8}$$

Those neighborhoods satisfy

$$\mathcal{V}(x) \subset B_x(\kappa(x)); \qquad \overline{\Gamma}(x) \subset B_x(\mathrm{C}'\kappa(x)). \tag{4.9}$$

4.2. *Travel times and radial limits.* We now come back to the spatial epidemic model. In this subsection, we estimate the time needed by the epidemic to cover C^∞, taking advantage of the analysis of paths in the percolation model done in Section 3. We first define an approximation for the passage time of the epidemic, then we prove the existence of radial limits for this approximation and for the epidemic. We will follow for this the spirit of the construction in Cox and Durrett (1988).

By analogy with Cox and Durrett (1981, 1988) (although neighborhoods in our context are defined differently), we define, for $x, y \in \mathbb{Z}^d$, the travel time from $\mathcal{V}(x)$ to $\mathcal{V}(y)$ and the time spent around x to be (remember (2.4))

$$\widehat{\tau}(x,y) = \inf_{x' \in \mathcal{V}(x), y' \in \mathcal{V}(y)} \tau(x',y'); \tag{4.10}$$

$$u(x) = \begin{cases} \displaystyle\sum_{(y',z') \in \overline{\Gamma}(x)} \tau(y',z') & \text{if } \overline{\Gamma}(x) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases} \tag{4.11}$$

By Remarks 4.1, 4.7*(a)*, $\widehat{\tau}(x,y)$ is finite. If $\mathcal{V}(x) \cap \mathcal{V}(y) \neq \emptyset$, then $\widehat{\tau}(x,y) = 0$.

We now show that if $y \in C_x^{\text{out}} \setminus R_x^{\text{out}}$, $\widehat{\tau}(x,y)$ approximates $\tau(x,y)$.

Lemma 4.8. *For $x \in \mathbb{Z}^d$, if $y \in C_x^{\text{out}} \setminus R_x^{\text{out}}$, we have*

$$\widehat{\tau}(x,y) \leq \tau(x,y) \leq u(x) + \widehat{\tau}(x,y) + u(y). \tag{4.12}$$

Proof of Lemma 4.8. Let $\Gamma_{x,y}$ be an open path from x to y such that $\tau(x,y) = \overline{\tau}(\Gamma_{x,y})$. Since $y \notin R_x^{\text{out}}$ this path must intersect C^∞. Let c_1 and c_2 be the first and last points we encounter in C^∞ when moving from x to y along $\Gamma_{x,y}$. By Definition 4.4*(i)*, $c_1 \in \mathcal{V}(x)$ and $c_2 \in \mathcal{V}(y)$: indeed (for instance for c_1), either $x \in C^\infty$ and $c_1 = x$, or the point $a \in \partial B_x(\kappa(x)) \cap \Gamma_{x,y}$ does not belong to R_x^{out} and c_1 is the first point on $\Gamma_{x,y}$ between x and a; we might have $c_1 = c_2$, if $\mathcal{V}(x) \cap \mathcal{V}(y) \neq \emptyset$. We have, denoting by $\vee$ the concatenation of paths,

$$\Gamma_{x,y} = \Gamma_{x,c_1} \vee \Gamma_{c_1,c_2} \vee \Gamma_{c_2,y}$$

where Γ_{x,c_1} (resp. $\Gamma_{c_2,y}$) is an open path from x to c_1 contained in $B_x(\kappa(x))$ (resp. from c_2 to y contained in $B_y(\kappa(y))$) and Γ_{c_1,c_2} is an open path from c_1 to c_2. We then obtain the first inequality of (4.12) since:

$$\widehat{\tau}(x,y) \leq \overline{\tau}(\Gamma_{c_1,c_2}) \leq \overline{\tau}(\Gamma_{x,y}) = \tau(x,y).$$

To prove the second inequality of (4.12), let Γ_{d_1,d_2} be an open path from $d_1 \in \mathcal{V}(x)$ to $d_2 \in \mathcal{V}(y)$ such that $\overline{\tau}(\Gamma_{d_1,d_2}) = \widehat{\tau}(x,y)$. Since the open paths Γ_{x,c_1} from x to c_1 and $\Gamma^*_{c_1,d_1}$ (which exists by Remark 4.7*(b)*) from c_1 to d_1 have edges in $\overline{\Gamma}(x)$ (see (4.8)), the open path $\Gamma_{x,d_1} = \Gamma_{x,c_1} \vee \Gamma^*_{c_1,d_1}$ from x to d_1 satisfies $\overline{\tau}(\Gamma_{x,d_1}) \leq u(x)$.

Similarly, there is an open path $\Gamma_{d_2,y}$ from d_2 to y such that $\overline{\tau}(\Gamma_{d_2,y}) \leq u(y)$. We conclude with

$$\tau(x,y) \leq \overline{\tau}(\Gamma_{x,d_1}) + \overline{\tau}(\Gamma_{d_1,d_2}) + \overline{\tau}(\Gamma_{d_2,y}) \leq u(x) + \widehat{\tau}(x,y) + u(y).$$

□

We now prove that $\widehat{\tau}(.,.)$ is almost subadditive, which will enable us later on in Theorem 4.12 to appeal to Kingman's Theorem.

Lemma 4.9. *For all $x, y, z \in \mathbb{Z}^d$, we have the* subadditivity property

$$\widehat{\tau}(x,z) \leq \widehat{\tau}(x,y) + u(y) + \widehat{\tau}(y,z). \tag{4.13}$$

Proof of Lemma 4.9. Let $\Gamma_{a,b}$ be an open path from $a \in \mathcal{V}(x)$ to $b \in \mathcal{V}(y)$ such that $\widehat{\tau}(x,y) = \overline{\tau}(\Gamma_{a,b})$. Similarly, let $\Gamma_{c,d}$ be an open path from $c \in \mathcal{V}(y)$ to $d \in \mathcal{V}(z)$ such that $\widehat{\tau}(y,z) = \overline{\tau}(\Gamma_{c,d})$ (we might have $a = b$, $c = d$ or $b = c$). Since both b and c are in $\mathcal{V}(y)$ there exists an open path $\Gamma^*_{b,c}$ from b to c such that $\overline{\tau}(\Gamma^*_{b,c}) \leq u(y)$ (see Remark 4.7*(b)* and (4.8)). The lemma then follows since the concatenation of these three paths is an open path from a point of $\mathcal{V}(x)$ to a point of $\mathcal{V}(z)$ and

$$\widehat{\tau}(x,z) \leq \overline{\tau}(\Gamma_{a,b}) + \overline{\tau}(\Gamma^*_{b,c}) + \overline{\tau}(\Gamma_{c,d}) \leq \widehat{\tau}(x,y) + u(y) + \widehat{\tau}(y,z).$$

□

We introduce a new notation, for the length of the shortest path between two neighborhoods. For $x, y \in \mathbb{Z}^d$, let

$$\overline{D}(x,y) = \inf_{x' \in \mathcal{V}(x), y' \in \mathcal{V}(y)} D(x', y'). \tag{4.14}$$

Note that unlike $D(x,y)$, $\overline{D}(x,y)$ is always finite. Next proposition corresponds to Proposition 3.11*(i)* for $\overline{D}(x,y)$ instead of $D(x,y)$. It will be used in Lemma 4.11 which follows.

Proposition 4.10. *There exist constants* C_4 *and* $\alpha_4 > 0$ *such that*

$$P(\overline{D}(x,y) \geq \mathrm{C}_4\|x-y\|_1 + n) \leq \exp(-\alpha_4 n^{1/d}), \qquad \forall\, x, y \in \mathbb{Z}^d, n \in \mathbb{N}.$$

Proof of Proposition 4.10. Let C_2 be as in Lemma 3.10 and Proposition 3.11. Then

$$\begin{aligned}
& P(\overline{D}(x,y) \geq \mathrm{C}_2\|x-y\|_1 + (2d+1)\mathrm{C}_2 n) \\
\leq\ & P(\kappa(x) > n) + P(\kappa(y) > n) \\
+\ & P(\overline{D}(x,y) \geq \mathrm{C}_2\|x-y\|_1 + (2d+1)\mathrm{C}_2 n, \kappa(x) \leq n, \kappa(y) \leq n) \\
\leq\ & P(\kappa(x) > n) + P(\kappa(y) > n) \\
+\ & \sum_{x' \in B_x(n), y' \in B_y(n)} P(D(x',y') \geq \mathrm{C}_2\|x-y\|_1 + (2d+1)\mathrm{C}_2 n, x' \to y') \\
\leq\ & P(\kappa(x) > n) + P(\kappa(y) > n) \\
+\ & \sum_{x' \in B_x(n), y' \in B_y(n)} P(D(x',y') \geq \mathrm{C}_2\|x'-y'\|_1 + \mathrm{C}_2 n, x' \to y').
\end{aligned}$$

The result follows from Proposition 3.11 and Lemma 4.6. □

Of course, the random variables $u(x)$ and $\widehat{\tau}(x,y)$ are almost surely finite. But we will need later on repeatedly a better control of their size, provided by our next lemma.

Lemma 4.11. *For all $x, y \in \mathbb{Z}^d$, $r \in \mathbb{N} \setminus \{0\}$, $u(x)$ and $\widehat{\tau}(x,y)$ have a finite r-th moment.*

Proof of Lemma 4.11. By Lemma 4.6, $u(x)$ is bounded above by a sum of passage times $e(y,z)$ with y and z in the box $B_x(Y)$, where Y is a random variable whose moments are all finite. By Lemmas 4.6 and 4.10 the same happens to $\widehat{\tau}(x,y)$ (if $x' \in \mathcal{V}(x), y' \in \mathcal{V}(y)$ are the sites that achieve $\overline{D}(x,y)$, then $\widehat{\tau}(x,y) \le \tau(x',y')$). Therefore it suffices to show that if $(X_i, i \in \mathbb{N})$ is a sequence of i.i.d. random variables and N is a random variable taking values in $\mathbb{N}$, then the moments of $\sum_{i=1}^N X_i$ are all finite if it is the case for both the X_i's and N. To prove this write:

$$\begin{aligned} E(|\sum_{i=1}^N X_i|^r) &= \sum_{n=1}^\infty E(|X_1 + \cdots + X_n|^r \mathbf{1}_{\{N=n\}}) \\ &\le \sum_{n=1}^\infty [E(|X_1 + \cdots + X_n|^{2r}) P(N=n)]^{1/2} \\ &\le \sum_{n=1}^\infty [E(|X_1| + \cdots + |X_n|)^{2r} P(N=n)]^{1/2} \\ &\le \sum_{n=1}^\infty [n^{2r} \mathrm{C}_{2r} P(N=n)]^{1/2} \end{aligned}$$

where the second line comes from Cauchy-Schwartz' inequality, the factor n^{2r} counts the number of terms in the development of $(|X_1| + \cdots + |X_n|)^{2r}$ and the constant C_{2r} depends on the distribution of the X_i's. As N has all its moments finite $P(N=n)$ decreases faster than n^{-2r-4} and the sum is finite. □

We now construct a process $(\vartheta.)$ which will be subadditive in every direction, and will have a.s., by Kingman's Theorem, a radial limit denoted by μ. We will then check that $\widehat{\tau}(o,\cdot)$ also has, in every direction, the same radial limit, and we will extend this conclusion to $\tau(o,\cdot)$ on the set C_o^{out} of sites that have ever been infected. Hence we first prove

Theorem 4.12. *For all $z \in \mathbb{Z}^d$, there exists $\mu(z) \in \mathbb{R}^+$ such that almost surely*

$$\lim_{n\to+\infty} \frac{\widehat{\tau}(o,nz)}{n} = \mu(z) \qquad \textit{and} \tag{4.15}$$

$$\lim_{n\to+\infty} \left[\frac{\tau(o,nz)}{n} - \mu(z)\right] \mathbf{1}_{\{nz \in C_o^{\mathrm{out}}\}} = 0. \tag{4.16}$$

Proof of Theorem 4.12. *(i)* For all $z \in \mathbb{Z}^d$, $(m,n) \in \mathbb{N}^2$, let

$$\vartheta_z(m,n) = \widehat{\tau}(mz,nz) + u(nz). \tag{4.17}$$

The process $(\vartheta_z(m,n))_{(m,n)\in\mathbb{N}^2}$ satisfies the hypotheses of Kingman' subadditive ergodic theorem (see Liggett, 2005, Theorem VI.2.6) by (4.13). Hence (noticing also that $\vartheta_z(0,n) = \vartheta_{nz}(0,1)$) there exists $\mu(z) \in \mathbb{R}^+$ such that

$$\lim_{n\to+\infty} \frac{1}{n}\vartheta_z(0,n) = \lim_{n\to+\infty} E\left(\frac{\vartheta_z(0,n)}{n}\right) = \lim_{n\to+\infty} E\left(\frac{\vartheta_{nz}(0,1)}{n}\right)$$

$$= \inf_{n\in\mathbb{N}} E\left(\frac{\vartheta_z(0,n)}{n}\right) = \inf_{n\in\mathbb{N}} E\left(\frac{\vartheta_{nz}(0,1)}{n}\right) = \mu(z) \quad (4.18)$$

a.s. and in L^1. Since the random variables $(u(z) : z \in \mathbb{Z}^d)$ are identically distributed, it follows from Lemma 4.11 and Chebychev's inequality that $\sum_{n=0}^{\infty} P(u(nz) > n\varepsilon) < +\infty$ for all $\varepsilon > 0$, so that by Borel-Cantelli's Lemma

$$\lim_{n\to+\infty} \frac{u(nz)}{n} = 0, \text{ a.s.} \quad (4.19)$$

Thus by (4.17), (4.18), (4.19) we have (4.15) for all $z \in \mathbb{Z}^d$.

(ii) Since R_o^{out} is a.s. finite, if $nz \in C_o^{\text{out}}$, then $nz \in C_o^{\text{out}} \setminus R_o^{\text{out}}$ for n large enough. Hence, from Lemma 4.8, for n large enough we have

$$\left|\frac{\tau(o,nz)}{n} - \mu(z)\right| \mathbf{1}_{\{nz\in C_o^{\text{out}}\setminus R_o^{\text{out}}\}} \leq \frac{u(o)+u(nz)}{n} + \left|\frac{\widehat{\tau}(o,nz)}{n} - \mu(z)\right|$$

and we conclude that (4.16) is satisfied by (4.19) and (4.15). □

4.3. *Extending* μ. We have proved the existence of a linear propagation speed in every direction of $\mathbb{Z}^d$. Now, to derive an asymptotic shape result, in particular for the approximating travel times $(\widehat{\tau}(x,y), x, y \in \mathbb{Z}^d)$, we need to extend μ from $\mathbb{Z}^d$ to a Lipschitz, convex and homogeneous function on $\mathbb{R}^d$. The asymptotic shape of the epidemic will be given by the convex set D defined in (4.20) below. As a first step, we prove properties of μ on $\mathbb{Z}^d$.

Lemma 4.13. *The function μ satisfies the following properties for all $x, y \in \mathbb{Z}^d$, $k \in \mathbb{N}$:*
(i) $\mu(x) = \lim_{n\to+\infty} E\left(\frac{\widehat{\tau}(o,nx)}{n}\right)$,
(ii) $\mu(x+y) \leq \mu(x) + \mu(y)$,
(iii) $\mu(x) = \mu(-x)$,
(iv) $\mu(\mathrm{e}_i) = \mu(\mathrm{e}_\ell)$, $\forall i, \ell \in \{1,\dots,d\}$,
(v) $\mu(kx) = k\mu(x)$,
(vi) $\mu(x) \leq \mu(\mathrm{e}_1)\|x\|_1$.

Proof of Lemma 4.13. Since $\vartheta_x(0,n) = \widehat{\tau}(o,nx) + u(nx)$, part *(i)* follows from (4.18) and (4.19). To prove part *(ii)* write:

$$\begin{aligned}
\mu(x+y) &= \lim_{n\to+\infty} E\left(\frac{\widehat{\tau}(o,n(x+y))}{n}\right) \\
&\leq \lim_{n\to+\infty} E\left(\frac{\widehat{\tau}(o,nx)}{n}\right) + \lim_{n\to+\infty} E\left(\frac{\widehat{\tau}(nx,n(x+y))}{n}\right) + \lim_{n\to+\infty} E\left(\frac{u(nx)}{n}\right) \\
&= \lim_{n\to+\infty} E\left(\frac{\widehat{\tau}(o,nx)}{n}\right) + \lim_{n\to+\infty} E\left(\frac{\widehat{\tau}(nx,n(x+y))}{n}\right) \\
&= \mu(x) + \mu(y),
\end{aligned}$$

where the first equality follows from part *(i)*, the inequality from (4.13), the second equality from (4.19) and the third one from part *(i)* and translation invariance of $\widehat{\tau}$.

Parts *(iii)–(iv)* follow immediately from part *(i)* and the corresponding properties of $\widehat{\tau}(o,x)$. To prove part *(v)* write:

$$\mu(kx) = \lim_{n\to+\infty} E\left(\frac{\vartheta_{nkx}(0,1)}{n}\right) = k \lim_{n\to+\infty} E\left(\frac{\vartheta_{nkx}(0,1)}{nk}\right) = k\mu(x),$$

where the first and third equalities follow from (4.18). Finally, part *(vi)* follows from parts *(ii)–(iv)*. □

Next corollary extends Lemma 4.13*(iii)–(iv)*.

Corollary 4.14. *For any permutation σ of $\{1,\cdots,d\}$, any $y=(y_1,y_2,\cdots,y_d)\in\mathbb{Z}^d$ and any choice of the signs $\pm$,*

$$\mu(\pm y_{\sigma(1)},\pm y_{\sigma(2)},\cdots,\pm y_{\sigma(d)}) = \mu(y_1,y_2,\cdots,y_d).$$

Proof of Corollary 4.14. Clearly $\widehat{\tau}(o,(y_1,y_2,\ldots,y_d))$ has the same distribution as $\widehat{\tau}(o,(\pm y_{\sigma(1)},\pm y_{\sigma(2)},\cdots,\pm y_{\sigma(d)}))$ for any choice of the signs and any permutation σ, hence the corollary follows from Lemma 4.13*(i)*. □

Lemma 4.15. *Let $\gamma^* = \mu(\mathrm{e}_1)$. Then γ^* is a Lipschitz constant for μ. For all $u,v\in\mathbb{Z}^d$ we have*

$$|\mu(u)-\mu(v)| \le \gamma^*\|u-v\|_1.$$

Proof of Lemma 4.15. Let $y=u-v$, $x=v$. We have

$$\mu(u)-\mu(v) = \mu(x+y)-\mu(x) \le \mu(y) = \mu(u-v) \le \mu(\mathrm{e}_1)\|u-v\|_1,$$

where the inequalities follow from Lemma 4.13*(ii)* and *(vi)*. Similarly, taking $x=u$, $y=v-x$ gives

$$\mu(v)-\mu(u) \le \mu(\mathrm{e}_1)\|v-u\|_1,$$

and the lemma follows. □

In a second step, we extend μ to $\mathbb{R}^d$ and we introduce the set D.

Proposition 4.16. *There exists an extension of μ to $\mathbb{R}^d$, which is Lipschitz with Lipschitz constant γ^* given by Lemma 4.15, convex and homogeneous on $\mathbb{R}^d$. Moreover, $\mu(x)=0$ if and only if $x=o$ and the set*

$$D = \{x\in\mathbb{R}^d : \mu(x)\le 1\} \tag{4.20}$$

is convex, bounded and contains an open ball centered at o.

Proof of Proposition 4.16. We start by extending μ to $\mathbb{Q}^d$. For $x\in\mathbb{Q}^d\setminus\{o\}$ let

$$N_x = \min\{k\ge 1, k\in\mathbb{N} : kx\in\mathbb{Z}^d\} \quad \text{and} \tag{4.21}$$

$$\mu(x) = \frac{\mu(N_x x)}{N_x}. \tag{4.22}$$

We now prove that this extension is homogeneous: let $\alpha\in\mathbb{Q}$ be positive and let $x\in\mathbb{Q}^d$, $x\ne o$. Then, there exist $k_1,k_2\in\mathbb{N}$ multiples of N_x and $N_{\alpha x}$ respectively, such that $k_1x, k_2\alpha x\in\mathbb{Z}^d$ and $k_1x = k_2\alpha x$. Write

$$\mu(\alpha x) = \frac{\mu(N_{\alpha x}\alpha x)}{N_{\alpha x}} = \frac{\mu(k_2\alpha x)}{k_2} = \frac{\mu(k_1 x)}{k_2} = \frac{k_1}{k_2}\frac{\mu(k_1x)}{k_1} = \alpha\frac{\mu(N_x x)}{N_x} = \alpha\mu(x),$$

using (4.22) for the first equality, Lemma 4.13*(v)* for the second and fifth ones. To prove that μ is Lipschitz on $\mathbb{Q}^d$, let $x, y \in \mathbb{Q}^d \setminus \{o\}$. Then,

$$\begin{aligned}
|\mu(x) - \mu(y)| &= \left|\frac{\mu(N_x x)}{N_x} - \frac{\mu(N_y y)}{N_y}\right| = \left|\frac{\mu(N_y N_x x)}{N_y N_x} - \frac{\mu(N_x N_y y)}{N_x N_y}\right| \\
&= \frac{|\mu(N_x N_y x) - \mu(N_x N_y y)|}{N_x N_y} \\
&\leq \frac{\gamma^* \|N_x N_y x - N_x N_y y\|_1}{N_x N_y} = \gamma^* \|x - y\|_1,
\end{aligned}$$

using Lemma 4.13*(v)* for the second equality and Lemma 4.15 for the inequality.

To prove that μ is convex on $\mathbb{Q}^d$, take $x, y \in \mathbb{Q}^d$ and $\alpha \in \mathbb{Q} \cap (0,1)$. Then let k_1, k_2 be elements in $\mathbb{N}$ such that $k_1\alpha \in \mathbb{N}$, $k_2 x \in \mathbb{Z}^d$, $k_2 y \in \mathbb{Z}^d$ and write:

$$\begin{aligned}
&\mu(\alpha x + (1-\alpha)y) = \lim_{n\to+\infty} E\Big(\frac{\widehat{\tau}(o, n\alpha x + n(1-\alpha)y)}{n}\Big) \\
&= \lim_{n\to+\infty} E\Big(\frac{\widehat{\tau}(o, nk_1\alpha k_2 x + nk_1(1-\alpha)k_2 y)}{nk_1k_2}\Big) \\
&\leq \lim_{n\to+\infty} E\Big(\frac{\widehat{\tau}(o, nk_1\alpha k_2 x) + \widehat{\tau}(o, nk_1(1-\alpha)k_2 y) + u(nk_1\alpha k_2 x)}{nk_1k_2}\Big) \\
&= \lim_{n\to+\infty} E\Big(\frac{\widehat{\tau}(o, nk_1\alpha k_2 x) + \widehat{\tau}(o, nk_1(1-\alpha)k_2 y)}{nk_1k_2}\Big) \\
&= \frac{\mu(k_1k_2\alpha x) + \mu(k_1k_2(1-\alpha)y)}{k_1k_2} \\
&= \alpha\mu(x) + (1-\alpha)\mu(y),
\end{aligned}$$

where the first equality follows from Lemma 4.13*(i)*, the inequality from Lemma 4.9, the third equality from (4.19), the fourth from Lemma 4.13*(i)* and the last one from the homogeneity of μ on $\mathbb{Q}^d$.

Because μ is homogeneous, Lipschitz and convex on $\mathbb{Q}^d$, we can extend μ by continuity to $\mathbb{R}^d$.

To prove that $\mu(x) > 0$ if $x \neq o$ we argue by contradiction: assume $\mu(x) = 0$ and without loss of generality that $x = (x_1, \ldots, x_d)$ with $x_1 \neq 0$. First note that since μ is Lipschitz and homogeneous, the conclusion of Corollary 4.14 also holds for any $(x_1, \ldots, x_d) \in \mathbb{R}^d$, then write

$$\begin{aligned}
\mu(2x_1, 0, \cdots, 0) &= \mu(2x_1, 0, \cdots, 0) - \mu(x_1, x_2, \cdots, x_d) \\
&\leq \mu(x_1, -x_2, \cdots, -x_d) = 0,
\end{aligned}$$

using Lemma 4.13*(ii)* for the inequality, and Corollary 4.14 with the assumption $\mu(x) = 0$ for the last equality. Then since μ is homogeneous we get $\mu(\mathrm{e}_1) = 0$. However, considering a standard first passage percolation model with passage times $e(z, y)$ and adding a 'tilde' to quantities associated to this model, we have $\widetilde{\tau}(o, z) \leq \tau(o, z)$ a.s. for all $z \in \mathbb{Z}^d$. Since by Kesten (1986, Theorem (2.18)),

$$\lim_{n\to+\infty} \widetilde{\tau}(o, n\mathrm{e}_1) = \widetilde{\mu}(\mathrm{e}_1),$$

it follows from (4.16) that $\widetilde{\mu}(\mathrm{e}_1) \leq \mu(\mathrm{e}_1) = 0$. But from Kesten (1986, Theorems (1.7) and (1.15)) we get $\widetilde{\mu}(\mathrm{e}_1) > 0$, thus reaching a contradiction.

The convexity of μ implies that D is convex. We prove by contradiction that D contains an open ball centered at o: otherwise, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n \notin D$, $\lim_{n\to+\infty} x_n = 0$; therefore on the one hand $\mu(x_n) > 1$, and on the other hand $\lim_{n\to+\infty} \mu(x_n) = 0$ because $\mu(o) = 0$ and μ is continuous, hence a contradiction.

Finally we argue again by contradiction to prove that the set D is bounded: otherwise there would exist a sequence $(y_n)_{n\in\mathbb{N}}$ with $y_n \in D$ and $\|y_n\|_1 > n$. Then $x_n = y_n/\|y_n\|_1$ satisfies $\|x_n\|_1 = 1$, and, since μ is homogeneous, $\lim_{n\to+\infty} \mu(x_n) = 0$. By compactness $(x_n)_{n\in\mathbb{N}}$ has a converging subsequence to some x such that $\mu(x) = 0$ with $\|x\|_1 = 1$; since we have already proved there is no such x we get a contradiction. $\square$

4.4. *Behavior of* $\widehat{\tau}$. Our next result establishes how $\widehat{\tau}(o,z)$ grows for $z \in \mathbb{Z}^d$.

Theorem 4.17. *There exist $K = K(\lambda, d) > 0$ and $\alpha > 0$ such that*

$$\begin{aligned} P(\widehat{\tau}(o,z) > K\|z\|_\infty) &\le \exp(-\alpha(\|z\|_\infty^{1/d}), \quad \forall\, z \in \mathbb{Z}^d, \\ P(\widehat{\tau}(o,z) > K(\|z\|_\infty + n)) &\le \exp(-\alpha n^{1/d}), \quad \forall\, z \in \mathbb{Z}^d, n \in \mathbb{N}, \\ \sum_{z\in\mathbb{Z}^d} P(\widehat{\tau}(o,z) > K\|z\|_\infty) &< +\infty. \end{aligned}$$

Proof of Theorem 4.17. Let $K \ge 0, z \in \mathbb{Z}^d$ and let $\mathcal{B} = B(o,(\|z\|_\infty + n)/4) \times B(z,(\|z\|_\infty + n)/4)$. Then write:

$$\begin{aligned} & P(\widehat{\tau}(o,z) > K(\|z\|_\infty + n)) \\ \le\ & P(4\kappa(z) > \|z\|_\infty + n) + P(4\kappa(o) > \|z\|_\infty + n) + P(A) \end{aligned} \tag{4.23}$$

where

$$\begin{aligned} A &= \{\widehat{\tau}(o,z) > K(\|z\|_\infty + n), 4\kappa(z) \le \|z\|_\infty + n, 4\kappa(o) \le \|z\|_\infty + n\} \\ &\subset \cup_{(x,y)\in\mathcal{B}}\{x \to y, \tau(x,y) > K(\|z\|_\infty + n)\}. \end{aligned} \tag{4.24}$$

Note that if $(x,y) \in \mathcal{B}$ we have

$$\begin{aligned} \|z\|_\infty - n &\le 2\|x-y\|_\infty \le 3\|z\|_\infty + n \quad \text{and} \\ 3(\|z\|_\infty + n) &= 3\|z\|_\infty + n + 2n \ge 2(\|x-y\|_\infty + n). \end{aligned} \tag{4.25}$$

From (4.24), (4.25), for C_2 given in Proposition 3.11, we get:

$$\begin{aligned} P(A) \le & \sum_{x\in B(o,(\|z\|_\infty+n)/4)} \ \sum_{y\in B(z,(\|z\|_\infty+n)/4)} \\ & \Big(P(3\tau(x,y) > 2K(\|x-y\|_\infty + n), D(x,y) < (\mathrm{C}_2+1)(\|x-y\|_1 + n)) \\ & \quad + P(x \to y, D(x,y) \ge (\mathrm{C}_2+1)(\|x-y\|_1 + n))\Big). \end{aligned} \tag{4.26}$$

It now follows from Proposition 3.11 *(i)* that we have, for some $\alpha_5 > 0$,

$$\begin{aligned} & P(x \to y, D(x,y) \ge (\mathrm{C}_2+1)(\|x-y\|_1 + n)) \\ \le\ & \exp(-\alpha_5(\|x-y\|_1 + n)^{1/d}) \\ \le\ & \exp(-\alpha_5(\|x-y\|_\infty + n)^{1/d}). \end{aligned} \tag{4.27}$$

Then, taking K large enough, by large deviation results for exponential variables, we also have, for some $\alpha_6 > 0$,

$$P(3\tau(x,y) > 2K(\|x-y\|_\infty + n), D(x,y) < (\mathrm{C}_2+1)(\|x-y\|_1 + n))$$

$$\begin{aligned}
&\leq \quad P(3\tau(x,y) > 2K(\|x-y\|_\infty + n), D(x,y) < (\mathrm{C}_2+1)d(\|x-y\|_\infty + n)) \\
&\leq \quad \exp(-\alpha_6(\|x-y\|_\infty + n)). \qquad (4.28)
\end{aligned}$$

Hence, from (4.25)–(4.28), for some constants R and $\alpha_7 > 0$ we have:

$$P(A) \leq R(\|z\|_\infty + n)^{2d} \exp(-\alpha_7(\|z\|_\infty + n)^{1/d}),$$

which gives, by modifying the constants,

$$P(A) \leq R' \exp(-\alpha_8(\|z\|_\infty + n)^{1/d}). \qquad (4.29)$$

The theorem's statements now follow from (4.29), (4.23) and Lemma 4.6. □

4.5. *Asymptotic shape for $\widehat{\tau}$.* Next theorem is the last necessary step to prove the shape theorem.

Theorem 4.18. *Let $\varepsilon > 0$, and $\widehat{A}_t = \{z \in \mathbb{Z}^d : \widehat{\tau}(o,z) \leq t\}$. Then, a.s. for t large enough, for D defined in (4.20),*

$$(1-\varepsilon)tD \cap \mathbb{Z}^d \subset \widehat{A}_t \subset (1+\varepsilon)tD \cap \mathbb{Z}^d. \qquad (4.30)$$

In the sequel K and α are fixed constants satisfying the conclusions of Theorem 4.17, γ^* is the Lipschitz constant of μ (see Lemma 4.15) and N_x was defined in (4.21) for any $x \in \mathbb{Q}^d \setminus \{o\}$. To prove Theorem 4.18 we need the two following lemmas.

Lemma 4.19. *Let $\rho > 0$ and let $\delta \leq \rho/(2K)$. Then, for all $x \in \mathbb{Q}^d \setminus \{o\}$,*

$$\sum_{k>0} P(\sup_{z \in B_{kN_x x}(\delta k N_x) \cap \mathbb{Z}^d} \widehat{\tau}(kN_x x, z) \geq kN_x\rho) \quad < \quad \infty, \qquad (4.31)$$

$$\sum_{k>0} P(\sup_{z \in B_{kN_x x}(\delta k N_x) \cap \mathbb{Z}^d} \widehat{\tau}(z, kN_x x) \geq kN_x\rho) \quad < \quad \infty. \qquad (4.32)$$

Proof of Lemma 4.19. We derive only (4.31), since the proof of (4.32) is analogous. By translation invariance

$$P(\sup_{z \in B_{kN_x x}(\delta k N_x) \cap \mathbb{Z}^d} \widehat{\tau}(kN_x x, z) \geq kN_x\rho) = P(\sup_{z \in B(\delta k N_x) \cap \mathbb{Z}^d} \widehat{\tau}(o, z) \geq kN_x\rho).$$

Hence it suffices to show that

$$\sum_{k>0} P(\sup_{z \in B(\delta k N_x) \cap \mathbb{Z}^d} \widehat{\tau}(o, z) \geq kN_x\rho) < \infty.$$

Let $k > 0, z \in B(\delta k N_x) \cap \mathbb{Z}^d$. By Theorem 4.17 we have:

$$\begin{aligned}
P(\widehat{\tau}(o,z) \geq kN_x\rho) \quad &\leq \quad P(\widehat{\tau}(o,z) \geq K\|z\|_\infty + kN_x\rho/2) \\
&\leq \quad \exp\Big(-\alpha\Big\lfloor \frac{kN_x\rho}{2K} \Big\rfloor^{1/d}\Big).
\end{aligned}$$

Therefore, for some constant C,

$$\begin{aligned}
\sum_{k>0} P(\sup_{z \in B(\delta k N_x) \cap \mathbb{Z}^d} \widehat{\tau}(o, z) \geq kN_x\rho) \quad &\leq \quad \sum_{k>0} \mathrm{C}(\delta k N_x)^d \exp\Big(-\alpha\Big\lfloor \frac{kN_x\rho}{2K} \Big\rfloor^{1/d}\Big) \\
&< \quad \infty.
\end{aligned}$$

□

For $x \in \mathbb{Q}^d \setminus \{o\}$, $\delta > 0$, we define the cone associated to x of amplitude δ as

$$C_x(\delta) = \mathbb{Z}^d \cap \Big(\cup_{t \geq 0} B_{tx}(\delta t)\Big). \qquad (4.33)$$

Lemma 4.20. *Let $x \in \mathbb{Q}^d \setminus \{o\}$. Then for any $0 < \delta' < \delta$ the set $C_x(\delta') \setminus \cup_{k\geq 0} B_{kN_x x}(\delta k N_x)$ is finite.*

Proof of Lemma 4.20. Let

$$t_0 = \frac{N_x \|x\|_1}{\delta - \delta'}.$$

Since $\mathbb{Z}^d \cap \Big(\cup_{t\geq 0}^{t_0} B_{tx}(\delta' t)\Big)$ is finite, it suffices to show that

$$\cup_{t\geq t_0}^{\infty} B_{tx}(\delta' t) \subset \cup_{k\geq 0} B_{kN_x x}(\delta k N_x).$$

To prove this, pick $z \in B_{t_1 x}(\delta' t_1)$ for some $t_1 \geq t_0$. Let $k_0 = \inf\{i \in \mathbb{N} : iN_x \geq t_1\}$. Hence $0 \leq k_0 N_x - t_1 < N_x$, and

$$\begin{aligned} \|z - k_0 N_x x\|_1 &\leq \|z - t_1 x\|_1 + |t_1 - k_0 N_x| \|x\|_1 \\ &< \|z - t_1 x\|_1 + N_x \|x\|_1 \leq \delta' t_1 + N_x \|x\|_1 \\ &= \delta' t_1 + (\delta - \delta') t_0 \leq \delta t_1 \leq \delta k_0 N_x. \end{aligned}$$

Therefore $z \in B_{k_0 N_x x}(\delta k_0 N_x)$ and the lemma is proved. □

In the next proof we use that since the Lipschitz constant of μ is γ^* for the norm $\|.\|_1$ (by Proposition 4.16), it is $\gamma = \gamma^* d$ for the norm $\|.\|_\infty$.

Proof of Theorem 4.18. Fix $\varepsilon \in (0, 1)$ and let ρ, δ and ι be three small positive parameters such that $\delta \leq \rho/(2K)$, whose values will be determined later. The set $\mathcal{Y} = \{x \in \mathbb{Q}^d : 1 - 2\iota < \mu(x) < 1 - \iota\}$ is a ring between two balls with the same center but with a different radius, because by Proposition 4.16, μ is homogeneous and positive except that $\mu(o) = 0$. Hence the (compact) closure of $\mathcal{Y}$, which is recovered by balls of the same radius centered on the rational points of $\mathcal{Y}$, is in fact covered by a finite number of such balls. Thus there exists a finite subset Y of $\mathcal{Y}$ such that $\mathbb{Z}^d \subset \cup_{x\in Y} C_x(\delta/2)$ (if the balls recover the ring, the cones associated to them recover the whole space). Hence, to prove the first inclusion of (4.30) it suffices to show that for any $x \in Y$ and any sequences $(t_n)_{n>0}$ and $(z_n)_{n>0}$ such that $t_n \uparrow \infty$ in $\mathbb{R}^+$, $z_n \in C_x(\delta/2) \cap \mathbb{Z}^d$ with $\|z_n\|_\infty \geq n$ and $\mu(z_n) \leq (1 - \varepsilon)t_n$, we have $\widehat{\tau}(o, z_n) \leq t_n$ a.s. for n sufficiently large. So, let $(t_n)_{n>0}$ and $(z_n)_{n>0}$ be such sequences. Using Lemma 4.20 (taking a subsequence if necessary) let $k_n \in \mathbb{N}$ be such that $z_n \in B_{k_n N_x x}(\delta k_n N_x)$, hence $k_n \geq Cn$ for some constant C. Since μ is Lipschitz, write

$$k_n N_x (1 - 2\iota) \leq \mu(k_n N_x x) \leq \mu(z_n) + \gamma \delta k_n N_x \leq (1 - \varepsilon) t_n + \gamma \delta k_n N_x,$$

so that

$$k_n N_x \leq \Big(\frac{1 - \varepsilon}{1 - 2\iota - \gamma\delta}\Big) t_n. \tag{4.34}$$

It now follows from (4.34) and the subadditivity property (4.13) of $\widehat{\tau}$ that:

$$\frac{\widehat{\tau}(o, z_n)}{t_n} \leq \Big(\frac{1 - \varepsilon}{1 - 2\iota - \gamma\delta}\Big)\Big(\frac{\widehat{\tau}(o, k_n N_x x)}{k_n N_x} + \frac{u(k_n N_x x)}{k_n N_x} + \frac{\widehat{\tau}(k_n N_x x, z_n)}{k_n N_x}\Big).$$

Therefore, by Theorem 4.12, Lemma 4.11 (the variables $u(.)$ are identically distributed, and $k_n \geq Cn$), Lemmas 4.16 and 4.19 we obtain:

$$\limsup_{n\to+\infty} \frac{\widehat{\tau}(o, z_n)}{t_n} \leq \Big(\frac{1 - \varepsilon}{1 - 2\iota - \gamma\delta}\Big)\Big(\mu(x) + \rho\Big) \qquad \text{a.s.}$$

Since $x \in Y$ this implies:

$$\limsup_{n\to+\infty} \frac{\widehat{\tau}(o,z_n)}{t_n} \leq \Big(\frac{1-\varepsilon}{1-2\iota-\gamma\delta}\Big)\Big(1-\iota+\rho\Big) \qquad \text{a.s.}$$

Taking ι, ρ and δ small enough, the right hand side is strictly less than 1 which proves that $\widehat{\tau}(o,z_n) \leq t_n$ a.s. for n sufficiently large.

Similarly, to prove the second inclusion of (4.30) it suffices to show that for any $x \in Y$ and any sequences $t_n \uparrow \infty$ in $\mathbb{R}^+$ and z_n in $C_x(\delta/2) \cap \mathbb{Z}^d$ such that $\mu(z_n) \geq (1+\varepsilon)t_n$ we have $\widehat{\tau}(o,z_n) > t_n$ a.s. for n sufficiently large. As before, taking subsequences if necessary, we let $(t_n)_{n>0}$ and $(z_n)_{n>0}$ be such sequences, and $k_n \in \mathbb{N}$ be such that $z_n \in B_{k_nN_xx}(\delta k_n N_x)$. Proceeding then as for the first inclusion, we get:

$$k_nN_x(1-\iota) \geq \mu(k_nN_xx) \geq \mu(z_n) - \gamma\delta k_nN_x \geq (1+\varepsilon)t_n - \gamma\delta k_nN_x,$$

$$k_nN_x \geq \Big(\frac{1+\varepsilon}{1-\iota+\gamma\delta}\Big)t_n,$$

$$\frac{\widehat{\tau}(o,z_n)}{t_n} \geq \Big(\frac{1+\varepsilon}{1-\iota+\gamma\delta}\Big)\Big(\frac{\widehat{\tau}(o,k_nN_xx)}{k_nN_x} - \frac{u(z_n)}{k_nN_x} - \frac{\widehat{\tau}(z_n,k_nN_xx)}{k_nN_x}\Big),$$

and

$$\begin{aligned}\liminf_{n\to+\infty} \frac{\widehat{\tau}(o,z_n)}{t_n} &\geq \Big(\frac{1+\varepsilon}{1-\iota+\gamma\delta}\Big)\Big(\mu(x)-\rho\Big) \qquad \text{a.s.}\\ &\geq \Big(\frac{1+\varepsilon}{1-\iota+\gamma\delta}\Big)\Big(1-2\iota-\rho\Big) \qquad \text{a.s.}\end{aligned}$$

Now, taking ι, ρ and δ small enough, the right hand side is strictly bigger than 1 and the second inclusion of (4.30) is proved. □

4.6. *Asymptotic shape for the epidemic.* We can now prove our main result, the shape theorem.

Proof of Theorem 2.2. Let $\varepsilon > 0$ be given.

(i) We first show that the infection grows at least linearly as t goes to infinity, that is,

$$P\Big(\big(\Upsilon_t \cup \Xi_t\big) \supset \big((1-\varepsilon)tD \cap C_o^{\text{out}}\big) \text{ for all } t \text{ large enough}\Big) = 1. \tag{4.35}$$

Since R_o^{out} is finite a.s. this will follow from:

$$P\Big(\big(\Upsilon_t \cup \Xi_t\big) \supset \big((1-\varepsilon)tD \cap (C_o^{\text{out}} \setminus R_o^{\text{out}})\big) \text{ for all } t \text{ large enough}\Big) = 1. \tag{4.36}$$

By Theorem 4.18, if $0 < a < b$ then $atD \cap \mathbb{Z}^d \subset \widehat{A}_{bt}$ a.s. for t large enough. Hence, for t large enough $z \in (1-\varepsilon)tD \cap (C_o^{\text{out}} \setminus R_o^{\text{out}})$ implies

$$\widehat{\tau}(o,z) \leq (1-\varepsilon/2)t \text{ a.s.} \tag{4.37}$$

and by Lemma 4.8, $\tau(o,z) \leq (1-\varepsilon/2)t + u(o) + u(z)$. Since $u(o) < +\infty$ a.s. we have $u(o) < (\varepsilon/4)t$ a.s. for t large enough. Hence, by (2.7), (4.36) will follow if we show that $\sup_{z\in tD} u(z) \leq (\varepsilon/4)t$ a.s. for t large enough, which is implied by $\sup_{z\in(n+1)D} u(z) \leq (\varepsilon/4)n$ a.s. for $n = \lfloor t \rfloor$. By Proposition 4.16, D is bounded, hence the number of points in $(n+1)D$ with coordinates in $\mathbb{Z}$ is less than $\mathrm{C}_5(n+1)^d$ for some constant C_5. Then write

$$P\left(\sup_{z\in(n+1)D} u(z) \geq \frac{\varepsilon n}{4}\right) \quad \leq \quad \mathrm{C}_5(n+1)^d P\left(u(o) \geq \frac{\varepsilon n}{4}\right)$$

$$\leq \quad C_5(n+1)^d \frac{4^{d+2}}{(\varepsilon n)^{d+2}} E(u(o)^{d+2}).$$

Thus, by Lemma 4.11, $\sum_{n\in\mathbb{N}} P(\sup_{z\in(n+1)D} u(z) \geq \varepsilon n/4) < \infty$, and (4.36) follows from Borel-Cantelli's Lemma.

(ii) Next we show that

$$P\Big(\big(\Upsilon_t \cup \Xi_t\big) \subset \big((1+\varepsilon)tD \cap C_o^{\text{out}}\big) \text{ for all } t \text{ large enough}\Big) = 1. \tag{4.38}$$

If z belongs to Ξ_t or Υ_t, then by (2.7) and Lemma 4.8, $\widehat{\tau}(o,z) \leq t$ for $z \in C_o^{\text{out}} \backslash R_o^{\text{out}}$, which implies $z \in (1+\varepsilon)tD$ for t large enough by Theorem 4.18. Since R_o^{out} is finite (4.38) follows.

Putting together (4.35) and (4.38) yields (2.11).

(iii) Finally, assuming $E(|T_z|^d) < \infty$, we show that

$$P(\Upsilon_t \cap (1-\varepsilon)tD = \emptyset \text{ for } t \text{ large enough}) = 1. \tag{4.39}$$

Let $z \in (1-\varepsilon)tD \cap C_o^{\text{out}}$, then, by (2.7), (4.35) and the same reasoning as for (4.37), we have $\tau(o,z) \leq (1-\varepsilon/2)t$ if t is large enough. Hence, (4.39) will follow if we show that $T_z \geq (\varepsilon/2)\tau(o,z)$ occurs only for a finite number of z's. Indeed otherwise $T_z \leq (\varepsilon/2)(1-\varepsilon/2)t$ so that $\tau(o,z) + T_z < t$ if t is large enough: it means that the infection has reached site z and the time of infection from z is over before time t, hence z has recovered by time t, that is $z \in \Xi_t, z \notin \Upsilon_t$.

But for $\delta = (2(1+\varepsilon)\sup_{x\in D} \|x\|_\infty)^{-1}$ (by Proposition 4.16, D is bounded), we have $\tau(o,z) \geq \delta\|z\|_\infty$ except for a finite number of z's. Because if z satisfies $\tau(o,z) < \delta\|z\|_\infty$, then by (2.7) and (4.38), for $\delta\|z\|_\infty$ larger than some t_0, we have $z \in \big(\Upsilon_{\delta\|z\|_\infty} \cup \Xi_{\delta\|z\|_\infty}\big) \subset (1+\varepsilon)\delta\|z\|_\infty D$, hence the contradiction $\|z\|_\infty \leq \|z\|_\infty/2$.

Therefore, it suffices to show that for any $\delta' > 0$ the event $\{T_z \geq \delta'\|z\|_\infty\}$ can only occur for a finite number of z's. This will follow from Borel-Cantelli's Lemma once we prove that $\sum_{z\in\mathbb{Z}^d} P(T_z \geq \delta'\|z\|_\infty) < \infty$. To do so we write, since the T_z's are identically distributed:

$$\sum_{z\in\mathbb{Z}^d} P(T_z \geq \delta'\|z\|_\infty) = \sum_{n\in\mathbb{N}} \sum_{z:\|z\|_\infty=n} P(T_z \geq \delta' n) \leq c \sum_{n\in\mathbb{N}} n^{d-1} P(T_o \geq \delta' n)$$

for some constant c, and this last series converges because T_o has a finite moment of order d. Putting together (2.11) and (4.39) yields (2.12). □

Appendix A.

In this appendix we prove Theorem 3.5, Lemma 3.7 and (4.2) in the proof of Lemma 4.3. These proofs rely on dynamic renormalisation techniques introduced in Barsky et al. (1991). In applying these techniques we follow Grimmett (1999, Chapter 7) and Grimmett and Marstrand (1990), but we introduce some modifications. In particular by considering some larger boxes we avoid using the sprinkling technique more than once on any given bond. Because of this we only need to consider two different values of the infection parameter and we do not need to introduce the updating functions of Grimmett (1999). To simplify the notation we write the proofs for $d = 3$, but their generalizations to higher dimensions presents no problems.

We introduce parameters whose values will be settled in Lemma A.9 below. We fix $\lambda' > \lambda_c$ and adopt the terminology of Grimmett (1999, Chapter 7). Nonetheless, we might change names of constants if this creates confusions with the rest of our paper. In the sequel n, m and N are positive integers such that

$$2m < n \qquad \text{and} \qquad N = n + m + 1. \tag{A.1}$$

We consider our percolation model on the slab $\mathbb{Z}^2 \times [-3N, 3N]$. Recall that we denote by $(\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$ the canonical basis of $\mathbb{Z}^3$. For $x = (x_1, x_2, x_3) \in \mathbb{Z}^3$ and $k \in \mathbb{N}$ such that $-3N + k \leq x_3 \leq 3N - k$ we recall that $B(k) = [-k, k]^3$ and $B_x(k) = x + [-k, k]^3$. We divide the face $F(n) = \{x : x \in \partial B(n), x_1 = n\}$ of $\partial B(n)$ in 4 quadrants:

$$\begin{aligned} T^{+,+}(n) &= \{x : x \in \partial B(n), x_1 = n, x_2 \geq 0, x_3 \geq 0\}, \\ T^{+,-}(n) &= \{x : x \in \partial B(n), x_1 = n, x_2 \geq 0, x_3 \leq 0\}, \\ T^{-,+}(n) &= \{x : x \in \partial B(n), x_1 = n, x_2 \leq 0, x_3 \geq 0\}, \\ T^{-,-}(n) &= \{x : x \in \partial B(n), x_1 = n, x_2 \leq 0, x_3 \leq 0\}. \end{aligned}$$

and, for any choice of $(i, j) \in \{+, -\}^2$, we define a box of thickness $2m+1$ composed of translates of the corresponding quadrant, by

$$T^{i,j}(m, n) = \cup_{\ell=1}^{2m+1} \{\ell \mathrm{e}_1 + T^{i,j}(n)\}. \tag{A.2}$$

For any of the above sets a subindex as y means we translate it by y.

Definition A.1. A seed is a translate of $B(m)$ such that all its edges are λ'-open.

We will be looking for oriented open paths starting in a seed inside $B(2N)$, and: either *(a)* contained in the union of boxes $B(3N) \cup B_{6N\mathrm{e}_1}(3N)$ and reaching a seed inside $B_{6N\mathrm{e}_1}(2N) \cap B_{8N\mathrm{e}_1}(2N)$; or *(b)* contained in the union of boxes $B(3N) \cup B_{6N\mathrm{e}_2}(3N)$ and reaching a seed inside $B_{6N\mathrm{e}_2}(2N) \cap B_{8N\mathrm{e}_2}(2N)$. We will construct those paths in Lemma A.7 below.

An important tool in Grimmett (1999, Chapter 7) is the *sprinkling technique*, which enables some bonds, that would be closed otherwise, to be independently open with a probability larger than some $\varepsilon' > 0$. We therefore need to find a way to proceed similarly, in spite of the fact that we work with dependent percolation. For this, it is convenient to define the processes for different values of the rate of propagation $\widetilde{\lambda}$ on our common probability space and compare them. Let $(e_1(x, y), x, y \in \mathbb{Z}^3)$ be a collection of independent exponential r.v.'s with parameter 1. Then let $e_{\widetilde{\lambda}}(x, y) = \widetilde{\lambda}^{-1} e_1(x, y)$, and

$$X_{\widetilde{\lambda}}(x, y) = \begin{cases} 1 & \text{if } e_{\widetilde{\lambda}}(x, y) < T_x; \\ 0 & \text{otherwise.} \end{cases} \tag{A.3}$$

We recall that the event $\{T_x > e_{\widetilde{\lambda}}(x, y)\}$ occurs if and only if the oriented bond (x, y) is $\widetilde{\lambda}$-open.

The following lemma implies that given $\lambda > \delta_1 > 0$, there exists $\iota > 0$ such that for any $\widetilde{\lambda}$ such that $\widetilde{\lambda} + \delta_1 < \lambda$ the random field $\{X_{\widetilde{\lambda}+\delta_1}(u, v) : u, v \in \mathbb{Z}^3\}$ is stochastically above the random field $\{\max\{X_{\widetilde{\lambda}}(u, v), Y(u, v)\} : u, v \in \mathbb{Z}^3\}$ where the random variables $Y(u,v)$ are i.i.d. Bernoulli with parameter ι and are independent of the random variables $X_{\widetilde{\lambda}}(u,v)$. This lemma justifies the use of the sprinkling technique needed to prove Lemmas A.5 and A.7 below.

Lemma A.2. *Assume $\lambda > \delta_1 > 0$. There exists $\iota > 0$ such that for any $\widetilde{\lambda} > 0$ such that $\widetilde{\lambda} + \delta_1 < \lambda$, and any $x, y \in \mathbb{Z}^3$, with $y \sim x$,*

$$\begin{aligned} P(X_{\widetilde{\lambda}+\delta_1}(x,y) = 1 \quad | \quad & X_{\widetilde{\lambda}+\delta_1}(u,v),\ u,v \in \mathbb{Z}^3,\ u \sim v, (u,v) \neq (x,y); \\ & X_{\widetilde{\lambda}}(u,v),\ u,v \in \mathbb{Z}^3,\ u \sim v) > \iota \quad a.s. \end{aligned}$$

Proof of Lemma A.2. Recall that $X_{\widetilde{\lambda}+\delta_1}(x,y)$ is independent of the random variables $X_{\widetilde{\lambda}+\delta_1}(u,v), X_{\widetilde{\lambda}}(u,v), u \neq x, v \sim u$, hence it suffices to show that

$$\begin{aligned} P(X_{\widetilde{\lambda}+\delta_1}(x,y) = 1 \quad | \quad & X_{\widetilde{\lambda}+\delta_1}(x,z),\ z \sim x, z \neq y; \\ & X_{\widetilde{\lambda}}(x,z), z \sim x) > \iota \quad a.s. \end{aligned} \tag{A.4}$$

Since we are now conditioning on a finite number of random variables taking only values 0 and 1, (A.4) will follow from:

$$\begin{aligned} P(X_{\widetilde{\lambda}+\delta_1}(x,y) = 1 \quad | \quad & X_{\widetilde{\lambda}+\delta_1}(x,z) = a_z, z \sim x, z \neq y; \\ & X_{\widetilde{\lambda}}(x,z) = b_z, z \sim x) > \iota \end{aligned} \tag{A.5}$$

for all choices a_z and b_z in $\{0,1\}$ with $b_z \leq a_z$.

To prove (A.5), we denote by $\mathcal{N}_x$ the union of all partitions of the set $\{z \in \mathbb{Z}^3 : z \sim x\}$ of neighbors of x into three disjoint sets called $\mathcal{N}_x^1, \mathcal{N}_x^0, \mathcal{N}_x^{0,1}$ such that $y \in \mathcal{N}_x^1 \cup \mathcal{N}_x^{0,1}$. Using the inequality $P(A|B) \geq P(A \cap B)$ for two events A, B, and taking an arbitrary partition in $\mathcal{N}_x$ gives

$$\begin{aligned} & P(X_{\widetilde{\lambda}+\delta_1}(x,y) = 1 \,|\, X_{\widetilde{\lambda}+\delta_1}(x,u) = 0,\ \forall u \in \mathcal{N}_x^0, \\ & \quad X_{\widetilde{\lambda}}(x,v) = 1,\ \forall v \in \mathcal{N}_x^1; X_{\widetilde{\lambda}}(x,w) = 0, X_{\widetilde{\lambda}+\delta_1}(x,w) = 1,\ \forall w \in \mathcal{N}_x^{0,1}) \\ & \geq P(X_{\widetilde{\lambda}+\delta_1}(x,y) = 1,\ X_{\widetilde{\lambda}+\delta_1}(x,u) = 0,\ \forall u \in \mathcal{N}_x^0, \\ & \quad X_{\widetilde{\lambda}}(x,v) = 1,\ \forall v \in \mathcal{N}_x^1; X_{\widetilde{\lambda}}(x,w) = 0, X_{\widetilde{\lambda}+\delta_1}(x,w) = 1,\ \forall w \in \mathcal{N}_x^{0,1}). \end{aligned}$$

Let $a > 0$ be such that $P(T_x \in [a, a+\gamma]) > 0$ for all $\gamma > 0$, and let $\delta_2 \in (0,a)$ be such that b defined by

$$b := \frac{(a+\delta_2)\widetilde{\lambda}}{\widetilde{\lambda} + \delta_1}$$

satisfies $b < a$. On the event

$$\begin{aligned} & \{T_x \in [a, a+\delta_2/2), e_{\widetilde{\lambda}}(x,w) \in [a+\delta_2/2, a+\delta_2),\ \forall w \in \mathcal{N}_x^{0,1}, \\ & e_{\widetilde{\lambda}}(x,v) \in [a-\delta_2/2, a),\ \forall v \in \mathcal{N}_x^1, \\ & e_{\widetilde{\lambda}+\delta_1}(x,u) \in [a+\delta_2/2, a+\delta_2),\ \forall u \in \mathcal{N}_x^0\}, \end{aligned} \tag{A.6}$$

for all sites v such that $v \in \mathcal{N}_x^1$ we have $e_{\widetilde{\lambda}}(x,v) < T_x$ hence $X_{\lambda'}(x,v) = 1$; for all sites u such that $u \in \mathcal{N}_x^0$ we have $T_x < e_{\lambda'+\delta_1}(x,u)$ hence $X_{\widetilde{\lambda}+\delta_1}(x,u) = 0$; and for all sites w such that $w \in \mathcal{N}_x^{0,1}$ we have $T_x < e_{\widetilde{\lambda}}(x,w)$ and

$$e_{\widetilde{\lambda}+\delta}(x,w) = \frac{\widetilde{\lambda}}{\widetilde{\lambda}+\delta_1} e_{\widetilde{\lambda}}(x,w) \leq \frac{\widetilde{\lambda}}{\widetilde{\lambda}+\delta_1}(a+\delta_2) = b < a \leq T_x$$

hence $X_{\widetilde{\lambda}}(u,w) = 0$ and $X_{\widetilde{\lambda+\delta_1}}(u,w) = 1$.

Since the probability $p(\mathcal{N}_x^0, \mathcal{N}_x^1, \mathcal{N}_x^{0,1})$ of the event (A.6) is strictly positive, we conclude the proof of (A.5) by taking

$$\iota = \inf_{\mathcal{N}_x} p(\mathcal{N}_x^0, \mathcal{N}_x^1, \mathcal{N}_x^{0,1}).$$

□

It is in view of this sprinkling procedure that we chose a propagation rate $\lambda' > \lambda_c$. As in Grimmett (1999, Section 7.2), we go on with two key geometrical lemmas. The first one, Lemma A.3, corresponds to Grimmett (1999, Lemma 7.9), with a very similar proof that we omit consequently. The second one, Lemma A.5, corresponds to Grimmett (1999, Lemma 7.17), that it generalizes in view of its applications for Theorem 3.5 and Lemma 3.7.

Lemma A.3. *If $\lambda_c < \lambda'$ and $\eta > 0$, then there exist integers $m = m(\lambda', \eta)$ and $n = n(\lambda', \eta)$ satisfying (A.1) and such that*

$$P\big(\textit{there exists a } \lambda'\textit{-open path in } B(n) \cup T^{i,j}(m,n) \textit{ from } B(m)$$
$$\textit{to a seed contained in } T^{i,j}(m,n)\big) > 1 - \eta,$$

for any choice of $(i,j) \in \{+,-\}^2$.

Notation A.4. Given a subset V of $\mathbb{Z}^3$, $\delta > 0$ and $x \in V$, we let $\sigma(x, V, \lambda', \delta)$ be the σ-algebra generated by the indicator functions of the following collection of events:

$$\{T_y > e_{\lambda'}(y,z) : y \in V, z \sim y\} \cup \{T_y > e_{\lambda'+\delta}(y,z) : y \in V \cap B_x(n)^c, z \sim y\}. \quad \text{(A.7)}$$

Note that when $V \cap B_x(n)^c = \emptyset$, $\sigma(x, V, \lambda', \delta)$ is simply the σ-algebra generated by the indicator functions of $\{T_y > e_{\lambda'}(y,z) : y \in V, z \sim y\}$, which we will denote by $\sigma(V, \lambda')$.

For A a subset of $\mathbb{Z}^3$, recall from (3.2) that $\Delta_v A$ denotes the exterior vertex boundary of A.

Lemma A.5. *If $\lambda_c < \lambda'$ and $\epsilon, \delta > 0$, there exists $m = m(\lambda', \epsilon, \delta)$ and $n = n(\lambda', \epsilon, \delta)$ satisfying (A.1) and with the following property:*
For any choice of $(i,j) \in \{+,-\}^2$, any $x \in \mathbb{Z}^3$, any set $L \subset \mathbb{Z}^3$ such that

$$B_x(m) \subset L \subset \mathbb{Z}^3 \setminus T_x^{i,j}(m,n) \quad \text{(A.8)}$$

and for any $\sigma(x, L, \lambda', \delta)$-measurable event H of strictly positive probability we have:

$$P(G^{i,j} | H) \geq 1 - \epsilon, \quad \text{(A.9)}$$

where

$$\begin{aligned} G^{i,j} \quad = \quad & \big\{\textit{there exists a path contained in } B_x(n) \cup T_x^{i,j}(m,n) \textit{ going from } L \\ & \textit{ to a seed contained in } T_x^{i,j}(m,n) \textit{ and such that} \\ & \textit{its first edge } (u,v) \textit{ with } u \in L, v \in \Delta_v L \textit{ is } (\lambda'+\delta)\textit{-open} \\ & \textit{and all its other edges are } \lambda'\textit{-open}\big\}. \end{aligned}$$

Note that since a λ'-open edge is also $(\lambda'+\delta)$-open, all the involved edges in the path in $G^{i,j}$ are $(\lambda'+\delta)$-open, but we do not know if the first edge of this path is λ'-open.

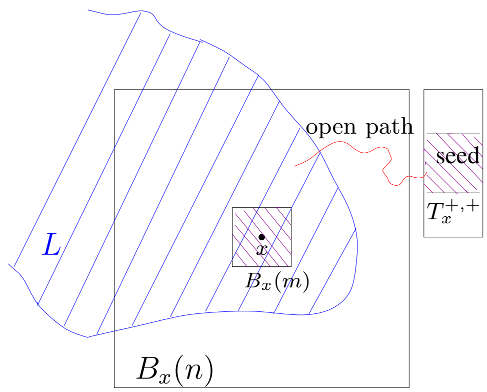

Figure A.2. In the open path, the first edge going out of L is $(\lambda'+\delta)$-open, and all other edges are λ'-open.

Proof of Lemma A.5. Given A a subset of $\mathbb{Z}^3$ and a subset C of $\Delta_v A$, let

$$\{A \overset{\lambda'+\delta}{\Rightarrow} C\} \tag{A.10}$$

be the event that at least one of the bonds going from A to C is $(\lambda'+\delta)$-open. Note that this is a stronger condition than to have a $(\lambda'+\delta)$-open path from A to C.

Let α be the probability that any given bond is λ'-open.

Since the model is invariant under translations and 90 degree rotations, it suffices to show the lemma when x is the origin and $(i,j)=(+,+)$. We will hence drop x, i and j from the notation. Let

$$\begin{aligned} V(L) = \big\{&z \in \Delta_v(L\cap B(n)) : \text{ there exists a } \lambda'\text{-open path contained in} \\ &B(n)\cup T(m,n)\setminus L \text{ going from } z \text{ to a seed contained in } T(m,n)\big\}. \end{aligned} \tag{A.11}$$

Here it is understood that there always is a λ'-open path from a point to itself. Therefore any $z\in\Delta_v(L\cap B(n))$ contained in a seed in $T(m,n)$ is in $V(L)$. Note also that since the path is contained in $B(n)\cup T(m,n)$, $V(L)$ is a subset of $B(n)\cup T(m,n)$. Now write

$$G = \cup_K \big(\{L \overset{\lambda'+\delta}{\Rightarrow} K\} \cap \{V(L)=K\}\big) \tag{A.12}$$

where the union is over all possible values of $V(L)$.

Our next step is to show that if m and n are properly chosen, the set $V(L)$ is large with probability close to 1. Let k be a positive integer. Note that if $V(L)$ has at most k elements, by the FKG inequality (see Remark 3.2), the probability that all the bonds entering $V(L)$ are λ'-closed is at least $(1-\alpha)^{6k}$ (recall that α is the probability that any given bond is λ'-open, and that we are working in $\mathbb{Z}^3$). All the λ'-open paths going from L to a seed contained in $T(m,n)$ have to pass through a point in $V(L)$. Hence if all the bonds entering $V(L)$ are λ'-closed, such a path does not exist. Thus we have

$$\begin{aligned} &P\big(\text{there exists a } \lambda'\text{-open path contained in } B(n)\cup T(m,n)\setminus L \\ &\quad \text{going from } L \text{ to a seed contained in } T(m,n) \mid |V(L)|\le k\big) \\ &\le 1-(1-\alpha)^{6k}. \end{aligned} \tag{A.13}$$

But according to Lemma A.3 there exist m and n such that $2m < n$ and

$$P\big(\text{there exists a } \lambda'\text{-open path contained in } B(n) \cup T(m,n) \text{ going from } B(m) \text{ to a seed contained in } T(m,n)\big)$$

is as close to 1 as we wish. This implies that

$$P\big(\text{there exists a } \lambda'\text{-open path contained (except its initial point) in } B(n) \cup T(m,n) \setminus L \text{ going from } L \text{ to a seed contained in } T(m,n)\big) \tag{A.14}$$

is as close to 1 as we wish uniformly in L's such that

$$B(m) \subset L \subset \mathbb{Z}^3 \setminus T(m,n). \tag{A.15}$$

The probability (A.14) is equal to

$$\begin{aligned}
&P\big(\text{there exists a } \lambda'\text{-open path contained (except its initial point) in} \\
&\quad B(n) \cup T(m,n) \setminus L \text{ going from } L \text{ to a seed contained in} \\
&\quad T(m,n) \,|\, |V(L)| \leq k\big) P(|V(L)| \leq k\big) \\
&+P\big(\text{there exists a } \lambda'\text{-open path contained (except its initial point) in} \\
&\quad B(n) \cup T(m,n) \setminus L \text{ going from } L \text{ to a seed contained in} \\
&\quad T(m,n) \,|\, |V(L)| > k\big) P(|V(L)| > k) \\
&\leq \big(1 - (1-\alpha)^{6k}\big) P(|V(L)| \leq k) + P(|V(L)| > k) \\
&= 1 - (1-\alpha)^{6k} P(|V(L)| \leq k),
\end{aligned} \tag{A.16}$$

where the inequality comes from (A.13). For this upper bound to be close to 1, $P(|V(L)| \leq k)$ has to be small. Hence, it follows from (A.13)–(A.16) that for any $\epsilon_0 > 0$ and any $k \in \mathbb{N}$, we can choose m and n with $2m < n$ in such a way that

$$P(|V(L)| \leq k) \leq \epsilon_0 \tag{A.17}$$

for all sets L satisfying (A.15).

Let K be a subset of $\Delta_v(L \cap B(n))$. We will now provide a lower bound to $P(L \overset{\lambda'+\delta}{\Rightarrow} K \,|\, H)$ which depends on the cardinality of K but is independent of H. Suppose K has at least $6r$ elements. Each point $u \in K$ has a neighbor $v \in L \cap B(n)$, that we associate to u. But since each point of $\mathbb{Z}^3$ has 6 nearest neighbors, v could be a neighbor of up to 6 points of K, to which it could have been associated. Then, there exist distinct $x_1, \ldots, x_r \in L \cap B(n)$ and distinct $y_1, \ldots, y_r \in K$ such that $x_i \sim y_i$ for $i = 1, \ldots, r$.

Since $x_i \in L \cap B(n)$ and H is $\sigma(o, L, \lambda', \delta)$-measurable, by Lemma A.2, for some $\iota > 0$ we have

$$P(X_{\lambda'+\delta}(x_i, y_i) = 0 \,|\, X_{\lambda'+\delta}(x_j, y_j) = 0, j = 1, \ldots i-1; H) < 1 - \iota \tag{A.18}$$

for $i = 1, \ldots, r$. It now follows from (A.18) and an inductive argument that

$$P(X_{\lambda'+\delta}(x_i, y_i) = 0,\, i = 1, \ldots, r \,|\, H) < (1-\iota)^r.$$

Hence for all $K \subset \Delta_v(L \cap B(n))$ such that $|K| \geq 6r$ we have:

$$P(L \overset{\lambda'+\delta}{\Rightarrow} K \,|\, H) \geq 1 - (1-\iota)^r. \tag{A.19}$$

Now write:

$$\begin{aligned} P(G, V(L)=K \mid H) &= P(L \overset{\lambda'+\delta}{\Rightarrow} K, V(L)=K \mid H) \\ &= P(L \overset{\lambda'+\delta}{\Rightarrow} K \mid V(L)=K, H) P(V(L)=K \mid H). \end{aligned}$$

But the event $\{V(L)=K\}$ is measurable with respect to the σ-algebra generated by the random variables $(T_x, e_{\lambda'}(x,y) : x \notin L)$ while both $\{L \overset{\lambda'+\delta}{\Rightarrow} K\}$ and H are measurable with respect to the σ-algebra generated by the random variables $(T_x, e_{\lambda'}(x,y), e_{\lambda'+\delta}(x,y) : x \in L)$. Therefore $\{V(L)=K\}$ is independent of the pair of events $H, \{L \overset{\lambda'+\delta}{\Rightarrow} K\}$, so that

$$P(G, V(L)=K \mid H) = P(L \overset{\lambda'+\delta}{\Rightarrow} K \mid H) P(V(L)=K)$$

Then summing up over all sets K such that $|K| \geq 6r$, it follows from (A.12) and (A.19) that

$$P(G|H) \geq (1-(1-\iota)^r) P(|V(L)| \geq 6r).$$

To complete the proof of Lemma A.5 first pick r such that $(1-\iota)^r < \epsilon/2$ and then use (A.17) to pick m and n such that $P(|V(L)| \geq 6r) \geq 1-\epsilon/2$. $\square$

Notation A.6. For a given $x \in \mathbb{Z}^3$ and $i = 1,2,3$, H_x^i will denote the hyperplane perpendicular to e_i passing through x. Before stating the next lemma where $x \in B(2N-m)$, we define $\overline{T}_x(m,n)$ to be the thickened box built from the quadrant opposite to the one x belongs to in the face $H_x^1 \cap B(2N-m)$, that is

$$\begin{array}{lll} T_x^{+,+}(m,n) & \text{if} & x_2 \leq 0 \text{ and } x_3 \leq 0, \\ T_x^{+,-}(m,n) & \text{if} & x_2 \leq 0 \text{ and } x_3 > 0, \\ T_x^{-,+}(m,n) & \text{if} & x_2 > 0 \text{ and } x_3 \leq 0, \\ T_x^{-,-}(m,n) & \text{if} & x_2 > 0 \text{ and } x_3 > 0. \end{array}$$

Thanks to Lemma A.5, in the following lemma we construct open paths starting in a seed inside $B(2N)$, and reaching either a seed inside $B_{6N\mathrm{e}_1}(2N) \cap B_{8N\mathrm{e}_1}(2N)$ or inside $B_{6N\mathrm{e}_2}(2N) \cap B_{8N\mathrm{e}_2}(2N)$. For $i \in \{1,2\}$, the successive seeds in these open paths will have centers belonging to the hyperplanes $H^i_{x+N\mathrm{e}_i}, H^i_{x+2N\mathrm{e}_i}, H^i_{x+3N\mathrm{e}_i}, H^i_{x+4N\mathrm{e}_i}, \ldots$; these successive seeds will be respectively contained in $B_{N\mathrm{e}_i}(2N)$, $B_{2N\mathrm{e}_i}(2N), B_{3N\mathrm{e}_i}(2N), B_{4N\mathrm{e}_i}(2N), \ldots$, and we will stop as soon as we will get a seed in $B_{8N\mathrm{e}_i}(2N)$. This construction will use a *steering procedure* in which, at each stage, the choice of a seed in $\overline{T}_{\cdot}(m,n)$ compensates from an earlier deviation.

Lemma A.7. *Given $\lambda' > \lambda_c$, for any $\epsilon, \delta > 0$ there exist $n = n(\lambda', \epsilon, \delta), m = m(\lambda', \epsilon, \delta)$ and N satisfying (A.1), such that for any $x \in B(2N-m)$,*

$$P(C_x^i) \geq 1 - 8\epsilon, \qquad \textit{for } i \in \{1,2\}$$

where

$$\begin{aligned} C_x^i = \big\{ & \textit{there exists a seed } B_y(m) \textit{ contained in } B_{8N\mathrm{e}_i}(2N), \textit{ with } y_i \leq 8N \\ & \textit{and a path contained in } B(3N) \cup B_{6N\mathrm{e}_i}(3N) \\ & \textit{from } B_x(m) \textit{ to } B_y(m) \textit{ whose edges are all } (\lambda'+\delta)\textit{-open and} \\ & \textit{those which are to the right of (resp. above) the hyperplane} \\ & H^i_{y-N\mathrm{e}_i} \textit{ when } i=1 \textit{ (resp. } i=2\textit{) are } \lambda'\textit{-open} \big\} \end{aligned}$$

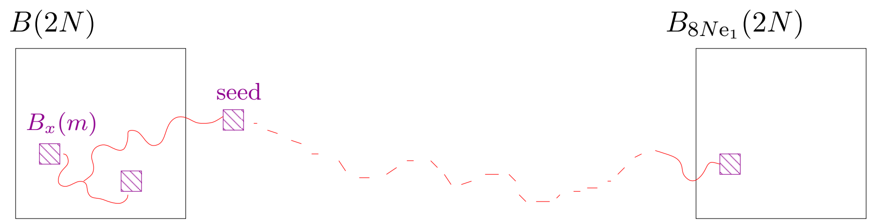

FIGURE A.3. Event C_x^1

Proof of Lemma A.7. We consider C_x^1. Since the model is invariant under 90 degree rotations, the proof will also be valid for C_x^2. Let V_1 be the set of vertices of all paths starting at $B_x(m)$ and contained in $B_x(n) \cup \overline{T}_x(m,n)$ whose first edge is $(\lambda' + \delta)$-open and all the other edges are λ'-open. Let

$$A_x^1 = \{V_1 \text{ contains a seed in } \overline{T}_x(m,n)\}.$$

Note that since $x \in B(2N-m)$ a path contained in $B_x(n) \cup \overline{T}_x(m,n)$ is also contained in $B(3N)$. Note also that the center of a seed in $\overline{T}_x(m,n)$ belongs to $H^1_{x+Ne_1}$, and that by our definition of $\overline{T}_x(m,n)$ (in Notation A.6) this seed is contained in $B_{Ne_1}(2N)$. Thanks to Lemma A.5 with $L = B_x(m)$ and H the whole probability space (that is without conditioning), there exist n, m such that

$$P(A_x^1) > 1 - \epsilon. \tag{A.20}$$

If A_x^1 occurs, of all the seeds in $\overline{T}_x(m,n) \cap V_1$ we choose one according to some arbitrary deterministic order. We now define a random variable Z_1 as follows: on A_x^1, Z_1 is the center of the chosen seed and on $(A_x^1)^c$, $Z_1 = \Delta$ where Δ is an extra point we add to $\mathbb{Z}^3$. Note that on A_x^1, Z_1 takes values in the hyperplane $H^1_{x+Ne_1}$. The random variable Z_1 is a function of V_1 which we denote by F_1. We now wish to give a lower bound to the conditional probability given $\{Z_1 = z_1\}$ with $z_1 \neq \Delta$ that there is a path contained in $B_{z_1}(n) \cup \overline{T}_{z_1}(m,n)$ from V_1 to a seed in $\overline{T}_{z_1}(m,n)$ and having the following properties:

(i) all its bonds are $(\lambda' + \delta)$-open;

(ii) all its bonds which are to the right of $H^1_{x+(N+m)e_1}$ are λ'-open.

Therefore to obtain the lower bound we let L_1 be a value of V_1 containing a seed in $\overline{T}_x(m,n)$ and consider the event:

$$\begin{aligned} A^1(x, L_1) \quad = \quad & \big\{\text{there exist } v_1 \in L_1 \cap B_{F_1(L_1)}(n) \text{ and a path} \\ & \text{from } v_1 \text{ to a seed in } \overline{T}_{F_1(L_1)}(m,n), \text{contained in} \\ & B_{F_1(L_1)}(n) \cup \overline{T}_{F_1(L_1)}(m,n) \text{ whose edges are all } (\lambda'+\delta)\text{-open} \\ & \text{and those to the right of } H^1_{x+(N+m)e_1} \text{ are } \lambda'\text{-open}\big\}. \end{aligned}$$

The event $\{V_1 = L_1\}$ is $\sigma(L_1, \lambda')$-measurable (recall Notation A.4), hence it follows from Lemma A.5 that

$$P(A^1(x, L_1) | V_1 = L_1) \geq 1 - \epsilon. \tag{A.21}$$

Let V_2 be the set of vertices of all the paths with the following properties:

(i) they start from $B_x(m)$;

(ii) they are contained in $B(3N) \cup B_{N\mathrm{e}_1}(3N)$ and lie entirely to the left of $H^1_{x+(2N+m)\mathrm{e}_1}$;

(iii) all their edges are $(\lambda'+\delta)$-open and those to the right of $H^1_{x+(N+m)\mathrm{e}_1}$ are λ'-open.

We also define the event

$$A^2_x = \{V_2 \text{ contains a seed centered in } H^1_{x+2N\mathrm{e}_1} \cap B_{2N\mathrm{e}_1}(2N-m)\}.$$

Noting that A^2_x contains $A^1(x,L_1) \cap \{V_1 = L_1\}$ for any L_1 containing a seed in $\overline{T}_x(m,n)$, summing over all such L_1's we get by (A.20) and (A.21)

$$\begin{aligned} P(A^2_x) &\geq \sum_{L_1} P(A^1(x,L_1)|V_1 = L_1)P(V_1 = L_1) \\ &\geq \sum_{L_1} (1-\epsilon)P(V_1 = L_1) = (1-\epsilon)P(A^1_x) \\ &\geq (1-\epsilon)^2 \geq 1-2\epsilon. \end{aligned} \tag{A.22}$$

Now we define a random variable Z_2 as follows: on the event A^2_x among the seeds contained in V_2 and centered in $H^1_{x+2N\mathrm{e}_1} \cap B_{2N\mathrm{e}_1}(2N-m)$ we choose one according to some arbitrary deterministic order and we let Z_2 be its center. On $(A^2_x)^c$ we let $Z_2 = \Delta$. Thus, Z_2 is a function F_2 of V_2. As before we let L_2 be a possible value of V_2 containing a seed centered in $H^1_{x+2N\mathrm{e}_1} \cap B_{2N\mathrm{e}_1}(2N-m)$ and consider the event:

$$\begin{aligned} A^2(x,L_2) = \big\{ & \text{there exist } v_2 \in L_2 \cap B_{F_2(L_2)}(n) \text{ and a path from } v_2 \text{ to a} \\ & \text{seed in } \overline{T}_{F_2(L_2)}(m,n), \text{ contained in } B_{F_2(L_2)}(n) \cup \overline{T}_{F_2(L_2)}(m,n) \\ & \text{whose edges are all } (\lambda'+\delta)\text{-open and those to the right of} \\ & H^1_{x+(2N+m)\mathrm{e}_1} \text{ are } \lambda'\text{-open}\big\}. \end{aligned}$$

The event $\{V_2 = L_2\}$ is $\sigma(F_2(L_2), L_2, \lambda', \delta)$-measurable, hence it follows from Lemma A.5 that

$$P(A^2(x,L_2)|V_2 = L_2) \geq 1-\epsilon.$$

We now let V_3 be the set of vertices belonging to all the paths with the following properties:

(i) they start from $B_x(m)$;

(ii) they are contained in $B(3N) \cup B_{N\mathrm{e}_1}(3N)$ and lie entirely to the left of $H^1_{x+(3N+m)\mathrm{e}_1}$;

(iii) all their edges are $(\lambda'+\delta)$-open and those to the right of $H^1_{x+(2N+m)\mathrm{e}_1}$ are λ'-open.

We also define the event:

$$A^3_x = \{V_3 \text{ contains a seed centered in } H^1_{x+3N\mathrm{e}_1} \cap B_{3N\mathrm{e}_1}(2N-m)\}.$$

Since A^3_x contains $A^2(x,L_2) \cap \{V_2 = L_2\}$ we can argue as before and get:

$$P(A^3_x) \geq 1-3\epsilon.$$

The argument is then repeated until we reach a seed in $B_{8N\mathrm{e}_1}(2N)$. The total number of steps needed is at most 8. Since at each step the probability is reduced by ϵ, the lemma is proved. □

Then define

$$C_x = C^1_x \cap C^2_x. \tag{A.23}$$

From Lemma A.7 we get:

Corollary A.8. *Given $\lambda' > \lambda_c$, for any $\epsilon, \delta > 0$ there exist n, m, N satisfying* (A.1) *and such that for any $x \in B(2N - m)$ we have*

$$P(C_x) \geq 1 - 16\epsilon.$$

Next lemma fixes the values of all the parameters introduced up to now.

Lemma A.9. *Assume $\lambda > \lambda_c$. Then, there exist constants m, N, K and $\iota > 0$ such that for all k,*

$$\begin{aligned}
&P(\textit{there exists a } \lambda\textit{-open path contained in} \\
&[-3N, (3+8k)N] \times [-3N, (3+8k)N] \times [-3N, 3N] \\
&\textit{from } B(m) \textit{ to a seed in } B_{8Nk\mathrm{e}_1+8Nk\mathrm{e}_2}(2N) \\
&\textit{whose number of edges is at most } 2Kk) \geq \iota.
\end{aligned}$$

Proof of Lemma A.9. We first fix $\epsilon > 0$ small enough for the two dimensional oriented site percolation of parameter $1-16\epsilon$ to be supercritical. Then we take $\lambda' > \lambda_c$ and $\delta > 0$ such that $\lambda' + \delta < \lambda$. Finally for those values of ϵ, δ and λ' we fix n, m and $N = n+m+1$ satisfying (A.1) and such that the conclusion of Corollary A.8 is valid.

We create a two dimensional oriented site percolation on $(\mathbb{Z}_+)^2$ associated to the percolation model we already have. We will refer to this model as the "renormalized model", while the percolation model we already had on $\mathbb{Z}^3$ will be referred to as the "original model". On the renormalized model all the paths are oriented upwards and towards the right; moreover, two subsequent sites of a path are at euclidean distance 1. We now explain the way in which these models are associated. In the renormalized model site $(0,0)$ is always considered open, site $(0,1)$ is open (closed) if C_0^1 occurs (does not occur) in the original model. Similarly, $(1,0)$ is open (closed) if C_0^2 occurs (does not occur) in the original model. Note that although the states of these last two sites $(0,1)$ and $(1,0)$ are dependent, by Corollary A.8 they are both open with probability at least $1-16\epsilon$. We then proceed recursively as follows.

At the n-th step we will look at the points in $\{(x,y) \in \mathbb{Z}_+^2 : x+y = n-1\}$ which have been reached in the renormalized model from $(0,0)$ following open paths and order them according to their second coordinates. We start from the point having the lowest second coordinate. Assume it is $(x_1, n-1-x_1)$. This point was reached from either $(x_1 - 1, n-1-x_1)$ or $(x_1, n-2-x_1)$. In the first case, in the original model a seed is reached in the left portion of $B_{8Nx_1\mathrm{e}_1+8N(n-1-x_1)\mathrm{e}_2}(2N)$ (remember the description given before the statement of Lemma A.7). Let z_1 be the center of this seed. If $C_{z_1}^1$ occurs (does not occur) in the original model we say that site $(x_1+1, n-1-x_1)$ in the renormalized model is open (closed). And if $C_{z_1}^2$ occurs (does not occur) in the original model we say that site $(x_1, n-x_1)$ is open (closed). Note that since z_1 is in the left portion of $B_{8Nx_1\mathrm{e}_1+8N(n-1-x_1)\mathrm{e}_2}(2N)$, when we attempt to move upwards, the first seed we are seeking is centered to the right of z_1 due to our steering procedure, thus avoiding regions where we have already used $(\lambda'+\delta)$-open edges. In the second case, the seed reached in the original model (we again denote its center by z_1) is in the lowest portion of $B_{8Nx_1\mathrm{e}_1+8N(n-1-x_1)\mathrm{e}_2}(2N)$ and when we want to establish if $C_{z_1}^1$ occurs we will be looking for paths reaching a seed whose center is above z_1. We then move to the second point in $\{(x,y) \in \mathbb{Z}_+^2 : x+y = n-1\}$ which has been reached in the renormalized model from $(0,0)$ following open paths. Let $(x_2, n-1-x_2)$ be that point and let z_2 be the center of the seed located inside $B_{8Nx_2\mathrm{e}_1+8N(n-1-x_2)\mathrm{e}_2}(2N)$ which was reached in the original model following open paths starting at $B(m)$. Two different cases arise: either $x_2 = x_1 - 1$ or $x_2 < x_1 - 1$. In the first case the point $(x_2+1, n-1-x_2) = (x_1, n-x_1)$ has already been declared open or closed and remains in that state. Then, we declare $(x_2, n-x_2)$ open (closed) if $C^2_{z_2}$ occurs (does not occur) in the original model. In the second case (when $x_2 < x_1 - 1$) we declare $(x_2+1, n-1-x_2)$ open (closed) if $C^1_{z_2}$ occurs (does not occur) in the original model and we declare $(x_2, n-x_2)$ open (closed) if $C^2_{z_2}$ occurs (does not occur) in the original model. Then we go on.

We now note that for all n each site examined in the set $\{(x,y) : x+y = n\}$ has probability bigger than $1-8\epsilon$ of being open and that such sites are dependent at most by pairs. This implies, as explained in the following lines, that the open cluster of the origin is stochastically above the open cluster of an independent oriented site percolation model of parameter $1-16\epsilon$.

For this, we again proceed by induction on n. We denote by $a_1, a_2, \ldots, a_k$ the points in the open cluster of the origin that belong to $\{(x,y) \in \mathbb{Z}_+^2 : x+y = n\}$. We assume that they are ordered according to their second coordinates. Point a_1 has two neighbors b_1, b_2 on $\{(x,y) \in \mathbb{Z}_+^2 : x+y = n+1\}$. They are both open with probability at least $1-16\epsilon$, which is stochastically larger than if they were both independently open with probability $1-16\epsilon$. In other words, if a random vector (Y_1, Y_2) with coordinates taking values in $\{0,1\}$ is such that $P(Y_1 = Y_2 = 1) \geq 1-16\epsilon$, then the vector (Y_1, Y_2) is stochastically larger than the vector (X_1, X_2) where X_1 and X_2 are independent Bernoulli r.v.'s of parameter $1-16\epsilon$. Going on, if $a_2 = a_1 + (-1,1)$, then we just have to consider the point $b_3 = a_2 + (0,1)$, because $a_2 + (1,0)$ has already been examined. This point b_3 will be open with probability at least $1-8\epsilon$ independently of what happened with b_1 and b_2. Otherwise if a_2 is more distant from a_1 we have to examine $b_3 = a_2 + (1,0)$ and $b_4 = a_2 + (1,0)$: they will both be open with probability at least $1-16\epsilon$ independently of what happened with b_1 and b_2, and so on. In the end, to each examined point on $\{(x,y) \in \mathbb{Z}_+^2 : x+y = n+1\}$ is attached a r.v. with value 1 if it is open and 0 if it is closed. The r.v.'s thus obtained are stochastically larger than a sequence of independent Bernoulli r.v.'s of parameter $1-16\epsilon$.

Thus, for our choice of ϵ the renormalized model is supercritical and there exists a constant $\iota > 0$ such that $P((0,0) \to (k,k)) \geq \iota$ for all $k \in \mathbb{N}$. Note also that the existence of an open oriented path from $(0,0)$ to (k,k) (which has length $2k$) in the renormalized model implies the existence of a $(\lambda' + \delta)$-open path in the original model from $B(m)$ to some seed in $B_{8Nk\mathrm{e}_1+8Nk\mathrm{e}_2}(2N)$ whose number of edges is bounded above by $2Kk$ where K is some constant that depends on N but not on k. Indeed suppose that the point following $(0,0)$ in the path of the renormalized model is $(1,0)$. This means that there exists an open path in the original model from a seed in $B(2N)$ to a seed in $B_{8N\mathrm{e}_1}(2N)$. This last path is not oriented, but being contained in $B(3N) \cup B_{6N\mathrm{e}_1}(3N)$, it uses only edges in this set. The total number of edges in the latter is a function of N which does not depend on k, that we denote by $K(N)$. Hence the derived open path in the original model from a seed in $B(2N)$ to a seed in $B_{8Nk\mathrm{e}_1+8Nk\mathrm{e}_2}(2N)$ has a number of edges bounded by $2K(N)k$. $\square$

For our next result we define the boxes:

$$\overline{B}_{i,j} = B_{(3+8i)N\mathrm{e}_1+(3+8j)N\mathrm{e}_2}(2N)$$

where i and j are non-negative integers.

Corollary A.10. *Assume $\lambda > \lambda_c$. Let N be as in the conclusion of Lemma A.9. Then, there exist $\iota' > 0$ and $K' \in \mathbb{N}$ such that: For any $k \in \mathbb{N}$ and any $0 \leq i_1, i_2, j_1, j_2 \leq k$ we have*

$$\begin{aligned}&P(\textit{there exists a } \lambda\textit{-open path contained in}\\ &[0,(6+8k)N] \times [0,(6+8k)N] \times [-3N,3N] \textit{ from } \overline{B}_{i_1,i_2} \textit{ to}\\ &\overline{B}_{j_1,j_2} \textit{ whose number of edges is at most } 2K'(|i_1-j_1|+|i_2-j_2|)) \geq \iota'.\end{aligned}$$

Proof of Corollary A.10. We wish to join $\overline{B}_{i_1,i_2}$ to $\overline{B}_{j_1,j_2}$. Lemma A.9 enables to go from a box to another one along a diagonal direction issued from that box. Hence applying Lemma A.9 we get

$$\begin{aligned}&P(\text{there exists a } \lambda\text{-open path contained in}\\ &[0,(6+8k)N] \times [0,(6+8k)N] \times [-3N,3N] \text{ from } \overline{B}_{i_1,i_2} \text{ to}\\ &\overline{B}_{i_1+r,i_2+r} \text{ whose number of edges is at most } 2Kr) \geq \iota\end{aligned}$$

for all $r \in \mathbb{N}$ such that $i_1 + r, i_2 + r \leq k$. Since the percolation model is invariant under 90 degree rotations, the same inequality holds if instead of adding (r,r) to (i_1,i_2) we add $(r,-r)$,$(-r,r)$ or $(-r,-r)$. That is, instead of going in one direction of one diagonal issued from (i_1,i_2), we may take this diagonal in the other direction, or one direction of the other diagonal issued from (i_1,i_2). This depends on the relative positions of (i_1,i_2) and (j_1,j_2) within the square $[0,k]\times[0,k]$ to which they both belong. More precisely, from (i_1,i_2) and from (j_1,j_2) is issued a diagonal, and those two diagonals intersect within $[0,k]\times[0,k]$. If this intersection point has integer coordinates, it can be written (i_1+r_1, i_2+r_2) as well as $(j_1+\ell_1, j_2+\ell_2)$, with $r_1 = r_2$ or $r_1 = -r_2$ (depending on which diagonal issued from (i_1,i_2) was used), and with $\ell_1 = \ell_2$ or $\ell_1 = -\ell_2$ similarly. If this intersection point does not have integer coordinates, on each of the involved diagonals there is one point with integer coordinates, with those two points at distance 1, of the form (i_1+r_1, i_2+r_2) and $(j_1+\ell_1, j_2+\ell_2)$, always with $r_1 = r_2$ or $r_1 = -r_2$, and $\ell_1 = \ell_2$ or $\ell_1 = -\ell_2$. To summarize, there exist integers r_1, r_2, ℓ_1, ℓ_2 with the following properties

(1) $r_2 = r_1$ or $r_2 = -r_1$ and $\ell_2 = \ell_1$ or $\ell_2 = -\ell_1$;
(2) $0 \leq i_1 + r_1, i_2 + r_2, j_1 + \ell_1, j_2 + \ell_2 \leq k$;
(3) either $|i_1 + r_1 - (j_1 + \ell_1)| + |i_2 + r_2 - (j_2 + \ell_2)| = 0$
or $|i_1 + r_1 - (j_1 + \ell_1)| + |i_2 + r_2 - (j_2 + \ell_2)| = 1$;
(4) $|r_1| + |\ell_1| \leq |i_1 - j_1| + |i_2 - j_2|$.

The corollary now follows from Lemma A.9, the FKG inequality (see Remark 3.2) and the fact that the distance from a point in $\overline{B}_{i_1+r_1,i_2+r_2}$ to a point in $\overline{B}_{j_1+\ell_1,j_2+\ell_2}$ is bounded above by $20N$. $\square$

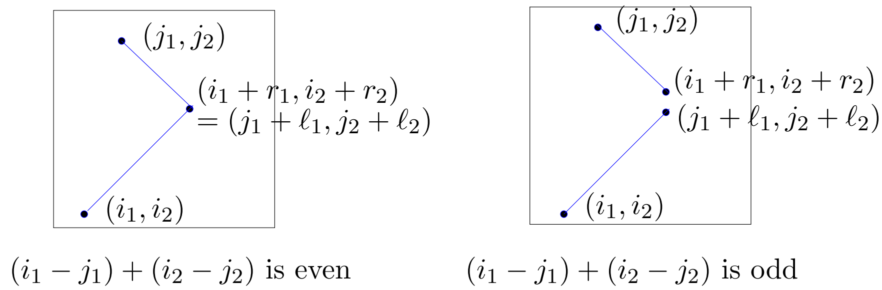

FIGURE A.4. Two possible cases

Proposition A.11. *Suppose $\lambda > \lambda_c$. Then there exist constants C, N and $\delta_1 > 0$ such that*

a) for all $M \geq 6N$, $x, y \in [0, M] \times [0, M] \times [-3N, 3N]$,

$$P\big(\text{there exists an open path from } x \text{ to } y \text{ contained in } [0, M] \times [0, M] \times [-3N, 3N] \text{ with at most } C\|x - y\|_1 \text{ edges}\big) \geq \delta_1$$

b) The original model is supercritical in a slab on thickness $k = 6N$.

Proof of Proposition A.11. It follows from Lemma A.9 that the probability of having an open path of length n starting in $B(m)$ and contained in the slab $\mathbb{Z} \times \mathbb{Z} \times [-3N, 3N]$ does not converge to 0 as n goes to infinity. This proves part *b)*. To prove part *a)* consider the boxes $B_{(3N+8Ni)e_1+(3N+8Nj)e_2}$ with $0 \leq i, j \leq (\frac{M}{N} - 6)\frac{1}{8}$. Then, note that for any point in $[0, M] \times [0, M] \times [-3N, 3N]$ there is such a box at distance at most $12N$. The result now follows from this, the FKG inequality (see Remark 3.2) and Corollary A.10. □

We have now all the ingredients for the proofs of Theorem 3.5, Lemma 3.7, and (4.2) of Lemma 4.3.

Proof of Theorem 3.5. Let x and y be two points in a slab of thickness $6N$. By Proposition A.11, the probability to have an open path from x to y in the slab is larger than δ_1. Therefore the probability for the outgoing cluster from x in the slab to be infinite, as well as the probability for the incoming cluster to y in the slab to be infinite, is at least δ_1.

Note that Proposition A.11 gives more precise information, since it restricts the involved open paths to a part of the slab, and gives an upper bound on the lengths of the paths. □

Proof of Lemma 3.7. For two points x and y, the idea to build an open path from x to y is to combine paths in different slabs using in each one Proposition A.11,*a)*.

(i) Let δ_1, M and C be given by Proposition A.11, and let $k \geq M$. For $n > 0$,

let $x = (x_1, x_2, x_3) \in B_{n+k} \setminus B_n$, $y = (y_1, y_2, y_3) \in (B_{n+k} \setminus B_n) \cup \Delta_v(B_{n+k} \setminus B_n)$. Assume for instance that $x_1 < -n$, $n < y_1$, $-n < x_2 < n$ and $-n < y_2 < n$. Let $u, v \in B_{n+k} \setminus B_n$ with $-n < u_1, n < u_2$ and $n < v_1, v_2$. By Proposition A.11,*a)* there exist with a probability larger than δ_1 an open path from x to u, as well as from u to v and from v to y. By FKG inequality (see Remark 3.2) there exists therefore with a probability larger than δ_1^3 an open path from x to y. Since this particular case gives the maximal distance between x and y, $\delta = \delta_1^3$ enables us to conclude.

(ii) Let $n < m$, $x \in A(n, m, 0), y \in A(n, m, 0) \cup \Delta_v A(n, m, 0)$. We proceed similarly to *(i)*. Assume for instance that $x_1 < n$, $x_2 < 0$ and $m < y_1$, $y_2 < 0$. Let $u, v \in A(n, m, 0)$ be such that $u_1 < n$, $0 < u_2$ and $m < v_1$, $0 < v_2$. By Proposition A.11,*a)* there exist with a probability larger than δ_1 an open path from x to u, as well as from u to v and from v to y. We conclude with $\delta = \delta_1^3$ and $C_1 = C$.

Note that we have to add $(-x_2)^+ + (-y_2)^+$ in part *(ii)* of the lemma because if $x \in \{z : -k + n \leq z_1 < n, -\infty < z_2 \leq 0\}$ and $y \in \{z : m < z_1 \leq m + k, -\infty < z_2 \leq 0\}$, to move from x to y staying in $A(n, m, 0)$ we need to reach first the set $\{z : -k + n \leq z_1 \leq m + k, 0 < z_2 \leq k\}$ (i.e. to increase the second coordinate until it is positive). □

Proof of (4.2) *of Lemma 4.3.* Relying on Proposition A.11,*b)*, we can follow the proof of Grimmett (1999, Theorems (8.18), (8.21)) to derive (4.2). □

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